Solutions to Review Problems for Final Exam

1. Let P_1 and P_2 denote two distinct points in $\mathbb{R}^3$. Let v_1 and v_2 denote two linearly independent vectors in $\mathbb{R}^3$. Let ℓ_1 denote the line through P_1 in the direction of v_1, and ℓ_2 denote the line through P_2 in the direction of v_2. Assuming that ℓ_1 and ℓ_2 do not meet, give a formula for computing the distance from ℓ_1 to ℓ_2.

   Solution: Let n denote the cross product of the vectors v_1 and v_2. Then, the plane, Γ, through P_1 and orthogonal to n contains the line ℓ_1. Since the vector v_2 is orthogonal to v_2 the line ℓ_2 is parallel to the plane. Hence, every point of the line ℓ_2 is at the same distance from the plane Γ. Hence,

   $$\begin{aligned} \operatorname{dist}(\ell_1, \ell_2) &= \operatorname{dist}(P_2, \Gamma) \\ &= \|\operatorname{Proj}_n(\overrightarrow{P_1P_2})\|, \end{aligned} \tag{1}$$

   where $\operatorname{Proj}_n(\overrightarrow{P_1P_2})$ is the orthogonal projection of the vector $\overrightarrow{P_1P_2}$ onto the direction of n; that is,

   $$\operatorname{Proj}_n(\overrightarrow{P_1P_2}) = \frac{\overrightarrow{P_1P_2} \cdot (v_1 \times v_2)}{\|v_1 \times v_2\|^2}(v_1 \times v_2). \tag{2}$$

   Combining the results of the calculations in (1) and (2), we get that

   $$\operatorname{dist}(\ell_1, \ell_2) = \frac{|\overrightarrow{P_1P_2} \cdot (v_1 \times v_2)|}{\|v_1 \times v_2\|}.$$

   □

2. In this problem, x and y denote vectors in $\mathbb{R}^n$.

   Let $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \sin(\|x\|)$, for all$x \in \mathbb{R}^n$. Prove that g is continuous on $\mathbb{R}^n$.

   Solution: Let $f(x) = \|x\|$ for all $x \in \mathbb{R}^n$ and observe that g is the composition of sin and f; that is,

   $$g(x) = (\sin \circ f)(x), \qquad \text{for all } x \in \mathbb{R}^n. \tag{3}$$

   Thus, the continuity of the g follows from that of sin and f. To see that $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous, first apply the triangle inequality to get that

   $$\|x\| \leqslant \|x - y\| + \|y\|, \qquad \text{for all } x, y \in \mathbb{R}^n,$$

from which we get that

$$\|x\| - \|y\| \leqslant \|x - y\|, \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{4}$$

Interchanging the roles for x and y in (4) we obtain

$$\|y\| - \|x\| \leqslant \|y - x\|.$$

from which we get

$$\|y\| - \|x\| \leqslant \|x - y\|. \tag{5}$$

Combining (4) and (5) yields

$$-\|x - y\| \leqslant \|x\| - \|y\| \leqslant \|x - y\|,$$

which implies that

$$|\|y\| - \|x\|| \leqslant \|y - x\|, \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{6}$$

It follows from (6) and the Squeeze Lemma that

$$\lim_{\|y-x\|\to 0} |f(y) - f(x)| = 0,$$

which shows that f is continuous at every $x \in \mathbb{R}^n$. It then follows from (3) and the continuity of sin that g is continuous on $\mathbb{R}^n$. □

3. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector v in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|v - t\widehat{u}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

   For any $v \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{u}$ which is the closest to v. Justify your answer.

   ***Solution**:* Use the properties of the dot product to compute

$$g(t) = \|v\|^2 - 2tv \cdot \widehat{u} + t^2, \tag{7}$$

since $\widehat{u}$ is a unit vector. It follows from (7) that $g(t)$ is a quadratic polynomial in t; hence, g is differentiable and

$$g'(t) = -2v \cdot \widehat{u} + 2t, \qquad \text{for all } t \in \mathbb{R}. \tag{8}$$

Observe that $g(t)$ gives the square of the distance from $t\widehat{u}$, an arbitrary element of the line spanned by $\widehat{u}$, to v. Thus, in order to find the point in span$\{\widehat{u}\}$ which is closest to v, we need to minimize g.

From (8) we get that

$$g''(t) = 2 > 0, \quad \text{for all } t \in \mathbb{R},$$

so that g has a global minimum when $g'(t) = 0$, or when $t = v \cdot \widehat{u}$. Thus, the point in span$\{\widehat{u}\}$ which is closest to v is $(v \cdot \widehat{u})\widehat{u}$, or the orthogonal projection of v onto $\widehat{u}$. □

4. Let f be a real valued function which is C^1 in an open interval containing the closed an bounded interval $[a, b]$. Define C to be the portion of the graph of f over $[a, b]$; that is,

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = f(x),\ a \leqslant x \leqslant b\}.$$

(a) Give a parametrization for C and compute the arc length, $\ell(C)$, of C.
***Solution**:* Let $\sigma \colon [a, b] \to \mathbb{R}^2$ be given by

$$\sigma(t) = (t, f(t)), \quad \text{for } t \in [a, b].$$

Then,

$$\sigma'(t) = (1, f'(t)), \quad \text{for } t \in (a, b),$$

so that

$$\|\sigma'(t)\| = \sqrt{1 + [f'(t)]^2}, \quad \text{for } t \in (a, b),$$

and, therefore, $\ell(C)$ is given by the formula

$$\ell(C) = \int_a^b \sqrt{1 + [f'(t)]^2}\ dt. \tag{9}$$

□

(b) Compute the arc length along the graph of $y = \ln x$ from $x = 1$ to $x = 2$.
***Solution**:* Apply the formula in (9) to compute

$$\begin{aligned} \ell(C) &= \int_1^2 \sqrt{1 + [\ln'(t)]^2}\ dt \\ &= \int_1^2 \sqrt{1 + \frac{1}{t^2}}\ dt, \end{aligned}$$

which can be written as

$$\ell(C) = \int_1^2 \frac{1}{t}\sqrt{t^2 + 1}\ dt. \tag{10}$$

Make the change of variables $u^2 = t^2 + 1$ in (10), so that

$$u\ du = t\ dt$$

and the integral in (10) now becomes

$$\ell(C) = \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2-1}\ du. \tag{11}$$

In order to evaluate the integral on the right–hand side of (11), first re–write the integrand as

$$\begin{aligned} \frac{u^2}{u^2-1} &= 1 + \frac{1}{u^2-1} \\ &= 1 + \frac{1}{(u+1)(u-1)}. \end{aligned} \tag{12}$$

Writing the last fraction in (12) as a sum of its partial fractions, we have

$$\frac{u^2}{u^2-1} = 1 + \frac{1/2}{u-1} - \frac{1/2}{u+1}. \tag{13}$$

Integrating with respect to u on both sides of (13) yields

$$\int \frac{u^2}{u^2-1}\ du = u + \frac{1}{2}\ln\left(\frac{|u-1|}{|u+1|}\right) + c, \tag{14}$$

for arbitrary constant c.
Next, use the integration formula in (14) to obtain from (11) that

$$\ell(C) = \sqrt{5} - \sqrt{2} + \frac{1}{2}\left[\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right].$$

□

5. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2}\ dydx$.

   (a) Identify the region of integration, R, for this integral and sketch it.
   ***Solution**:* The region $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant 1\}$ is sketched in Figure 1. □

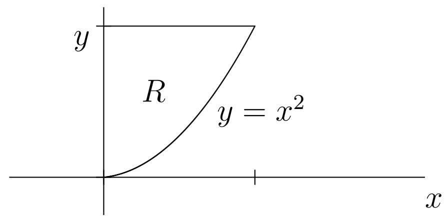

Figure 1: Sketch of Region R

(b) Change the order of integration in the iterated integral and evaluate the double integral $\int_R x\sqrt{1-y^2}\; dxdy$.

***Solution**:* Compute

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\; dxdy &= \int_0^1 \int_0^{\sqrt{y}} x\sqrt{1-y^2}\; dxdy \\ &= \int_0^1 \left[\frac{x^2}{2}\sqrt{1-y^2}\right]_0^{\sqrt{y}}\; dy \\ &= \int_0^1 \frac{y}{2}\sqrt{1-y^2}\; dy.\end{aligned}$$

Next, make the change of variables $u = 1 - y^2$ to obtain that

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\; dxdy &= -\frac{1}{4}\int_1^0 \sqrt{u}\; du \\ &= \frac{1}{4}\int_0^1 \sqrt{u}\; du \\ &= \frac{1}{6}.\end{aligned}$$

□

6. What is the region R over which you integrate when evaluating the iterated integral

$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}}\; \mathrm{d}y\; \mathrm{d}x?$$

Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

***Solution**:* The region $R = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant y \leqslant x,\ 1 \leqslant x \leqslant 2\}$ is sketched in Figure 2. Interchanging the order of integration, we obtain that

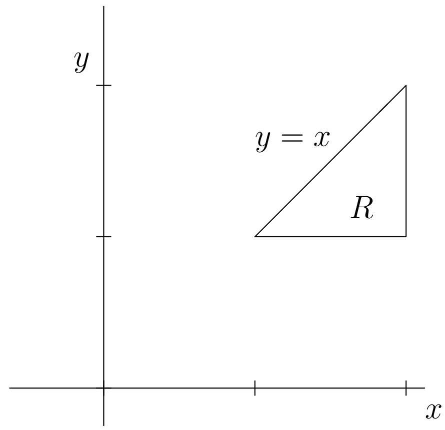

Figure 2: Sketch of Region R

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\, dxdy \quad = \quad \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2+y^2}}\, dxdy. \tag{15}$$

The iterated integral in (15) is easier to evaluate; in fact,

$$\begin{aligned} \iint_R \frac{x}{\sqrt{x^2+y^2}}\, dxdy &= \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2+y^2}}\, dxdy \\ &= \int_1^2 \left[\sqrt{x^2+y^2}\right]_y^2 dy \\ &= \int_1^2 \left[\sqrt{4+y^2} - \sqrt{2}\ y\right]\ dy. \end{aligned}$$

We therefore get that

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\, dxdy \quad = \quad \int_1^2 \sqrt{4+y^2}\ dy - \sqrt{2}\ \int_1^2 y\ dy. \tag{16}$$

Evaluating the second integral on the right–hand side of (16) yields

$$\int_1^2 y\ dy = \frac{3}{2}. \tag{17}$$

The first integral on the right–hand side of (16) leads to

$$\int_1^2 \sqrt{4+y^2}\ dy \quad = \quad \left[\frac{y}{2}\sqrt{4+y^2} + \frac{4}{2}\ln\left|y+\sqrt{4+y^2}\right|\right]_1^2,$$

which evaluates to

$$\int_1^2 \sqrt{4+y^2}\ dy \quad = \quad 2\sqrt{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right). \tag{18}$$

Substituting (17) and (18) into (16) we obtain

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\ dxdy \quad = \quad \frac{\sqrt{2}}{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right).$$

□

7. Let R denote the region in the xy–plane given by

$$R = \{(x,y)\in\mathbb{R}^2 \mid 0\leqslant x\leqslant 1,\ x^2\leqslant y\leqslant x\}.$$

Sketch a picture the region R and evaluate the line integral $\int_{\partial R} x^2\ \ \mathrm{d}x - xy\ \ \mathrm{d}y$, where ∂R is the boundary of R traversed in the counterclockwise sense.

***Solution**:* Apply Green's Theorem to get

$$\begin{aligned}\int_{\partial R} x^2\ \ \mathrm{d}x - xy\ \ \mathrm{d}y \quad &= \quad \iint_R \left(\frac{\partial}{\partial x}[-xy] - \frac{\partial}{\partial y}[x^2]\right) dxdy \\ &= \quad -\iint_R y\ dxdy\end{aligned} \tag{19}$$

We evaluate the double integral in (19) as the iterated integral

$$\begin{aligned}\iint_R y\ dxdy \quad &= \quad \int_0^1 \int_{x^2}^x y\ dydx \\ &= \quad \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^x\ dx,\end{aligned}$$

so that

$$\iint_R y \, dxdy \;=\; \frac{1}{2}\int_0^1 (x^2 - x^4)\, dx = \frac{1}{15}. \qquad (20)$$

Combining (19) and (20) yields

$$\int_{\partial R} x^2 \;\; \mathrm{d}x - xy \;\; \mathrm{d}y \;=\; -\frac{1}{15}.$$

□

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \;\; r = \sqrt{x^2+y^2} \quad \text{for all} \;\; (x,y) \in \mathbb{R}^2.$$

(a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.

***Solution**:* Compute

$$F(x,y) = \nabla u(x,y) = \frac{\partial u}{\partial x}\,\widehat{i} + \frac{\partial u}{\partial y}\,\widehat{j}, \qquad (21)$$

where, by the Chain Rule,

$$\frac{\partial u}{\partial x} = f'(r)\,\frac{\partial r}{\partial x} \qquad (22)$$

and

$$\frac{\partial u}{\partial y} = f'(r)\,\frac{\partial r}{\partial y}. \qquad (23)$$

In order to compute $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial r}{\partial x}$, write

$$r^2 = x^2 + y^2, \qquad (24)$$

and differentiate with respect to x on both sides of (24) to obtain

$$2r\frac{\partial r}{\partial x} = 2x,$$

from which we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \text{for } (x,y) \neq (0,0). \qquad (25)$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{for } (x, y) \neq (0, 0). \tag{26}$$

Substituting (25) into (22) yields

$$\frac{\partial u}{\partial x} = \frac{f'(r)}{r}\, x. \tag{27}$$

Similarly, substituting (26) into (23) yields

$$\frac{\partial u}{\partial y} = \frac{f'(r)}{r}\, y. \tag{28}$$

Next, substitute (27) and (28) into (21) to obtain

$$F(x, y) = \frac{f'(r)}{r}(x\,\widehat{\imath} + y\,\widehat{\jmath}), \tag{29}$$

□

(b) Recall that the divergence of a vector field $F = P\,\widehat{\imath} + Q\,\widehat{\jmath}$ is the scalar field given by $\text{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Express the divergence of the gradient of u, in terms of f', f'' and r.

The expression $\text{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.

Solution*:* From (29) we obtain that

$$P(x, y) = \frac{f'(r)}{r}\, x \quad \text{and} \quad Q(x, y) = \frac{f'(r)}{r}\, y,$$

so that, applying the Product Rule, Chain Rule and Quotient Rule,

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{f'(r)}{r} + x\frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial x} \\ &= \frac{f'(r)}{r} + x\,\frac{rf''(r) - f'(r)}{r^2}\,\frac{x}{r}, \end{aligned} \tag{30}$$

where we have also used (25). Simplifying the expression in (30) yields

$$\frac{\partial P}{\partial x} = \frac{f'(r)}{r} + x^2\,\frac{f''(r)}{r^2} - x^2\,\frac{f'(r)}{r^3}. \tag{31}$$

Similar calculations lead to

$$\frac{\partial Q}{\partial y} = \frac{f'(r)}{r} + y^2\ \frac{f''(r)}{r^2} - y^2\ \frac{f'(r)}{r^3}. \tag{32}$$

Adding the results in (31) and (32), we then obtain that

$$\begin{aligned} \text{div}F &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= 2\ \frac{f'(r)}{r} + r^2\ \frac{f''(r)}{r^2} - r^2\ \frac{f'(r)}{r^3}, \end{aligned} \tag{33}$$

where we have used (24). Simplifying the expression in (33), we get that

$$\text{div}F = f''(r) + \frac{f'(r)}{r}.$$

□