Assignment #16

Due on Friday, November 4, 2011

Read Section 4.9, *Solving the Logistic equation,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`, starting on page 64.

Read Section 4.9.2, *Partial Fractions,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`, starting on page 67.

Do the following problems

1. *Logistic Growth*[1]. Suppose that the growth of a certain animal population is governed by the differential equation

$$\frac{1000}{N}\frac{dN}{dt} = 100 - N,$$

where $N(t)$ denote the number of individuals in the population at time t.

(a) Suppose there are 200 individuals in the population at time $t = 0$. Sketch the graph of $N = N(t)$.

(b) Will there ever be more than 200 individuals in the population? Will there ever be fewer than 100 individuals? Explain your answer.

2. *Spread of a viral infection*[2]. Let $I(t)$ denote the total number of people infected with a virus. Assume that $I(t)$ grows according to a logistic model. Suppose that 10 people have the virus originally and that, in the early stages of the infection the number of infected people doubles every 3 days. It is also estimated that, in the long run 5000 people in a given area will become infected.

(a) Solve an appropriate logistic model to find a formula for computing $I(t)$, where t is the time from the initial infection measured in weeks. Sketch the graph of $I(t)$.

(b) Estimate the time when the rate of infected people begins to decrease.

[1] Adapted from Problem 6 on page 521 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002

[2] Adapted from Problem 7 on page 521 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002

3. *Non–Logistic Growth*[3]. There are many classes of organisms whose birth rate is not proportional to the population size. For example, suppose that each member of the population requires a partner for reproduction, and each member relies on chance encounters for meeting a mate. Assume that the expected number of encounters is proportional to the product of numbers of female and male members in the population, and that these are equally distributed; hence, the number of encounters will be proportional to the square of the size of the population.

   Use a conservation principle to derive the population model

   $$\frac{dN}{dt} = aN^2 - bN, \tag{1}$$

   where a and b are positive constants. Explain your reasoning.

4. For the equation in (1),

   (a) find the values of N for which the population size is not changing;

   (b) find the range of positive values of N for which the population size is increasing, and those for which it is decreasing;

   (c) find ranges of positive values of N for which the graph of $N = N(t)$ is concave up, and those for which it is concave down;

   (d) Sketch possible solutions.

5. For the equation in (1),

   (a) use separation of variables and partial fractions to find a solution satisfying the initial condition $N(0) = N_o$, for $N_o > 0$.

   (b) What happens to $N(t)$ as $t \to \infty$ if $N_o > b/a$? What happens if $N_o < b/a$? Why is b/a called a threshold value?

[3] Adapted from Problem 12 on page 39 in Braun, *Differential Equations and their Applications,* Fourth Edition, Springer–Verlag, 1993