Solutions Review Problems for Exam #1

1. Water leaks out a barrel at a rate proportional to the square root of the depth of the water at that time. If the water level starts at 36 inches and drops to 35 inches in 1 minute, how long will it take for the water to leak out of the barrel?

   ***Solution**:* Let $h = h(t)$ denote the water level in the barrel at time t, where h is measured in inches and t in minutes. We then have that

   $$\frac{dh}{dt} = -k\sqrt{h}, \tag{1}$$

   where k is a constant of proportionality.

   We can solve the equation in (1) by separating variables to obtain

   $$\int \frac{1}{\sqrt{h}}\, dh = -\int k\, dt,$$

   which integrates to

   $$2\sqrt{h} = -kt + c_1, \tag{2}$$

   where c_1 is an arbitrary constant. Dividing both sides of the equation in (2) by 2 and squaring, we obtain

   $$h(t) = \left(c - \frac{k}{2}t\right)^2, \tag{3}$$

   where we have set $c = c_1/2$.

   In order to find what c in (3) is, we use the information $h(0) = 36$ to obtain

   $$c^2 = 36,$$

   from which we obtain that $c = 6$, so that (3) now becomes

   $$h(t) = \left(6 - \frac{k}{2}t\right)^2. \tag{4}$$

   Next, use the information that $h(1) = 35$ to estimate the value of k in (4). We have that

   $$\left(6 - \frac{k}{2}\right)^2 = 35,$$

from which we obtain that

$$k = 2(6 - \sqrt{35}) \doteq 0.16784. \tag{5}$$

To find the time, t, at which all the water leaks out of the barrel, se solve the equation

$$h(t) = 0,$$

or

$$\left(6 - \frac{k}{2}t\right)^2 = 0,$$

to obtain that

$$t = \frac{12}{k}. \tag{6}$$

Using the estimate for k in (5), we obtain from (6) that

$$t \doteq 71.5 \ \text{ minutes.}$$

Thus, it will take about 1 hour and 11.5 minutes for the water to leak out of the barrel. □

2. The rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time. If an initial dose of Q_o is injected directly into the blood, 20% is left in the blood after 3 hours.

   (a) Write and solve a differential equation for the quantity, Q, of the drug in the blood at time, t, in hours.

   Solution: Apply the conservation principle

   $$\frac{dQ}{dt} = \text{Rate of substance in} - \text{Rate of substance out},$$

   where

   $$\text{Rate of substance in} = 0$$

   and

   $$\text{Rate of substance out} = kQ,$$

   where k is a constant of proportionality. Hence,

   $$\frac{dQ}{dt} = -kQ. \tag{7}$$

   □

(b) How much of the drug is left in the patient's body after 6 hours if the patient is given 100 mg initially?

Solution: The solution to the differential equation (7) subject to the initial condition $Q(0) = Q_o$ is given by

$$Q(t) = Q_o e^{-kt}, \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

To estimate the value of k, we use the information that $Q(3) = 0.2Q_o$ to obtain the equation

$$Q_o e^{-3k} = 0.2Q_o,$$

which can be solved for k to obtain

$$k = -\frac{\ln(0.2)}{3} \doteq 0.536479. \tag{9}$$

Next, use (8) to compute

$$Q(6) = Q_o e^{-6k}. \tag{10}$$

Putting $Q_o = 100$ mg, and using the estimate for k in (9), we obtain from (10) that

$$Q(6) \doteq 100e^{-6(0.54)} \doteq 4.0 \text{ mg}.$$

□

3. Use the Fundamental Theorem of Calculus to show that $y(t) = y_o \exp(F(t))$, where F is the antiderivative of f with $F(0) = 0$, is a solution to the initial value problem $\dfrac{dy}{dt} = f(t)y, \quad y(0) = y_o$.

Solution: Apply the Chain Rule to obtain

$$\begin{aligned} \frac{dy}{dt} &= y_0 \exp'(F(t))F'(t) \\ &= y_0 \exp(F(t))f(t) \\ &= f(t)[y_0 \exp(F(t))], \end{aligned}$$

which shows that

$$\frac{dy}{dt} = f(t)y.$$

Next, compute

$$y(0) = y_o \exp(F(0)) = y_o \exp(0) = y_o.$$

Hence, if $F\colon I \to \mathbb{R}$ is differentiable over some open interval I which contains 0, with $F' = f$ on I , and $F(0) = 0$, then $y(t) = y_o \exp(F(t))$ for $t \in I$ solves the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t)y \\ y(0) = y_o. \end{cases}$$

□

4. Find a solution to the initial value problem $\dfrac{dy}{dt} = e^{t-y}, \quad y(0) = 1.$

***Solution**:* Write the differential equation as

$$\frac{dy}{dt} = e^t e^{-y},$$

and separate variables to obtain

$$\int e^y \, dy = \int e^t \, dt,$$

which integrates to

$$e^y = e^t + c, \tag{11}$$

for arbitrary c. Using the initial condition $y(0) = 1$ in (11) yields

$$e = 1 + c,$$

from which we get that

$$c = e - 1. \tag{12}$$

Substituting the value for c in (12) into the equation in (11) yields

$$e^y = e^t + e - 1,$$

which can be solved for y to obtain

$$y(t) = \ln[e^t + e - 1], \quad \text{for all } t \in \mathbb{R}.$$

□

5. Evaluate the following integrals

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{2-e^{-x}}\,dx$ (b) $\displaystyle\int \frac{1}{x\ln x}\,dx$

(c) $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$ (d) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Solution:

(a) Make the change of variables $u = 2 - e^{-x}$, so that $du = e^{-x}\,dx$. Then,

$$\int_0^1 \frac{e^{-x}}{2-e^{-x}}\,dx = \int_1^{2-e^{-1}} \frac{1}{u}\,du = \ln(2-e^{-1}).$$

(b) Make the change of variables $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$ and

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + c \\ &= \ln|\ln x| + c.\end{aligned}$$

(c) Make the change of variables $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$ and

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du = \frac{1}{2}[\ln 2]^2.$$

(d) Make the change of variables $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$, and

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int e^u\,du \\ &= 2e^u + c \\ &= 2e^{\sqrt{x}} + c.\end{aligned}$$

□

6. The temperature in a hot iron decreases at a rate 0.11 times the difference between its present temperature and room temperature (20° C).

   (a) Write a differential equation for the temperature of the iron.

   Solution: Let $u = u(t)$ denote the temperature of the hot iron at time t. Then,

   $$\frac{du}{dt} = -0.11(u - 20), \tag{13}$$

   where u is measured in degrees Celsius and t in minutes. □

   (b) If the initial temperature of the rod is 100° C, and the time is measured in minutes, how long will it take for the rod to reach a temperature of 25° C?

   Solution: The general solution of the differential equation in (13) is

   $$u(t) = 20 + ce^{-0.11\ t}, \qquad \text{for all } t \in \mathbb{R}, \tag{14}$$

   for arbitrary constant c.
   To find the value of c in (14), we use the initial condition $u(0) = 100$ in (14) to obtain the equation

   $$20 + c = 100,$$

   which yields

   $$c = 80. \tag{15}$$

   Substituting the value of c in (15) into the expression for u in (14), we obtain that

   $$u(t) = 20 + 80e^{-0.11\ t}, \qquad \text{for all } t \in \mathbb{R}. \tag{16}$$

   Next, we find the value of t for which $u(t) = 25$, or

   $$20 + 80e^{-0.11\ t} = 25,$$

   or

   $$80e^{-0.11\ t} = 5,$$

   which can be solved for t to yield

   $$t = -\frac{\ln(1/16)}{0.11} = \frac{4\ln 2}{0.11} \doteq 25 \text{ minutes}.$$

   Thus, it will take about 25 minutes for the hot iron to reach the temperature or 25 degrees Celsius. □