Solutions to Exam 1

1. When people smoke, carbon monoxide is released into the air. Suppose that in a room of volume 60 m^3, air containing 5% carbon monoxide is introduced at a rate of 0.002 m^3/min. (This means that 5% of the volume of incoming air is carbon monoxide). The carbon monoxide mixes immediately with the air and the mixture leaves the room at the same rate as it enters.

   (a) Let $Q = Q(t)$ denote the volume (in cubic meters) of carbon monoxide in the room at any time t in minutes. Use a conservation principle to derive a differential equation for Q.

   Solution: Imagine the room as a compartment of, fixed volume, V. In this case, $V = 60$ m^3. Air flows into the room at rate, F, of 0.002 cubic meters per minute. The air that flows into the room has a concentration, c_i, of carbon monoxide, where $c_i = 5\%$ (the concentration here is measured in percent volume). Let $Q(t)$ denote the amount of carbon monoxide present in the room at time t. Apply the conservation principle

   $$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},$$

   where

   $$\text{Rate of } Q \text{ in} = c_i F,$$

   and

   $$\text{Rate of } Q \text{ out} = c(t) F,$$

   where $c(t) = \dfrac{Q(t)}{V}$ is the concentration of carbon monoxide in the room at time t. Here we are assuming that the volume, V, of air in the room is fixed, so that the rate of flow of air into the room is the same as the rate of flow out of the room.
   We then have that

   $$\frac{dQ}{dt} = c_i F - \frac{F}{V}\, Q.$$

   Putting in the values of F, V and c_i we obtain

   $$\frac{dQ}{dt} = 10^{-4} - \frac{1}{3} \times 10^{-4} Q, \tag{1}$$

   in units of cubic meters per minute. □

(b) Give the equilibrium solution, $\overline{Q}$, to the differential equation in part (a).

Solution: The equilibrium point, $\overline{Q}$,is the solution to the equation

$$10^{-4} - \frac{1}{3} \times 10^{-4} Q = 0,$$

which yields

$$\overline{Q} = 3 \text{ cubic meters.} \tag{2}$$

□

(c) Solve the differential equation in part (a) under the assumption that the there is no carbon monoxide in the room initially, and sketch the solution.

Solution: Rewrite the equation in (1) in the form

$$\frac{dQ}{dt} = -\frac{1}{3} \times 10^{-4}[Q - \overline{Q}],$$

or

$$\frac{dQ}{dt} = -k[Q - \overline{Q}], \tag{3}$$

where we have set

$$k = \frac{1}{3} \times 10^{-4}, \tag{4}$$

we see that the general solution of (1) is

$$Q(t) = \overline{Q} + ce^{-kt}, \quad \text{for } t \geqslant 0, \tag{5}$$

and some arbitrary constant c.
Using the condition $Q(0) = 0$ in (13) we obtain the equation

$$\overline{Q} + c = 0,$$

which yields $c = -\overline{Q}$. Substituting this value for c in (13) yields

$$Q(t) = \overline{Q}[1 - e^{-kt}], \quad \text{for } t \geqslant 0, \tag{6}$$

as a solution for (3) satisfying the initial condition $Q(0) = 0$. A sketch of the solution is shown in Figure 1. □

(d) Based on your solution to part (c), give the concentration, $c(t)$, of carbon monoxide in the room (in percent volume) at any time t in minutes. What happens to the value of $c(t)$ in the long run?

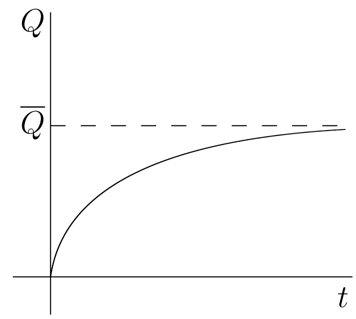

Figure 1: Sketch of graph of $Q(t)$

Solution: Divide the expression in (14) by V yields

$$c(t) = \frac{\overline{Q}}{V}[1 - e^{-kt}], \quad \text{for all } t \geqslant 0,$$

or

$$c(t) = 0.05[1 - e^{-kt}], \quad \text{for all } t \geqslant 0, \tag{7}$$

where k is as given in (4). It then follows from (7) that

$$\lim_{t\to\infty} c(t) = 0.05,$$

so that $c(t)$ tends towards 5% in the long run. □

(e) Medical texts warn that exposure to air containing 0.1% carbon monoxide for some time can lead to a coma. How many hours does it take for the concentration of carbon monoxide found in part (d) to reach this level?

Solution: We need to find the time, t, for which $c(t) = 0.01$. Using (7), we obtain the equation

$$0.05[1 - e^{-kt}] = 0.001,$$

which can be solved for t to yield

$$t = \frac{\ln(50) - \ln(49)}{k},$$

where k is as given in (4), so that

$$t \doteq 606 \text{ minutes},$$

or about 10 hours. □

2. Suppose that $y = y(t)$ is a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = e^{-t^2}, & t \in \mathbf{R}, \\ y(0) = 0. \end{cases} \tag{8}$$

   (a) Find y' and y''.

   Solution: The derivative of y is given by the differential equation in (8) as

$$y'(t) = e^{-t^2}, \quad \text{for all } t \in \mathbf{R}. \tag{9}$$

   Next, differentiate $y'(t)$ in (9) with respect to t to obtain

$$y''(t) = -2t\, e^{-t^2}, \quad \text{for all } t \in \mathbf{R}, \tag{10}$$

   where we have applied the Chain Rule. □

   (b) Determine the values of t for which $y(t)$ increases or decreases, and the values of t for which the graph of $y = y(t)$ is concave up or concave down. Sketch the graph of $y = y(t)$ given that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$.

   Solution: Since the exponential function is always positive, it follows from (9) that $y'(t) > 0$ for all $t \in \mathbf{R}$, so that $y(t)$ increases for all values of t.
   Similarly, we obtain from (10) $y''(t) > 0$ for $t < 0$, and $y''(t) < 0$ for $t > 0$; so that, the graph of $y = y(t)$ is concave up for $t < 0$ and concave down for $t > 0$.
   A sketch of the graph of $y = y(t)$ is shown in Figure 2. □

3. Assume that the relative growth rate of a certain animal population is governed by the equation

$$\frac{1}{N}\frac{dN}{dt} = k_o\, e^{-t}, \tag{11}$$

   where $N = N(t)$ is the number of individuals in the population t units of time from the time we start observing the population, and k_o is a positive constant.

   (a) Give an interpretation for this model and explain how it differs from the Malthus model for population growth.

   Solution: The equation in (11) models a situation in which the *per capita* growth rate decreases exponentially with time. This is to be contrasted with the Malthusian model for population growth in which the *per capita* growth rate remains constant. □

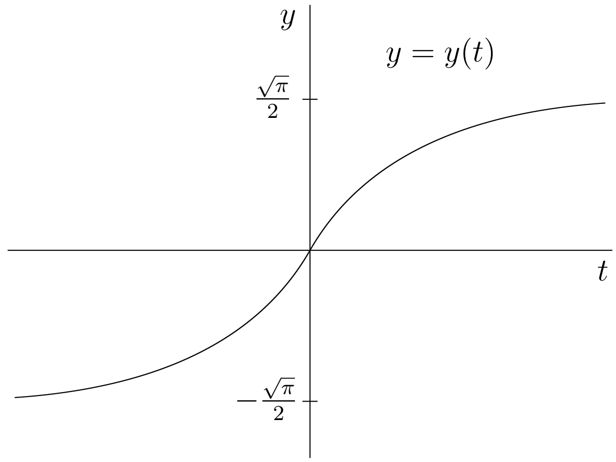

Figure 2: Sketch of graph of $y = y(t)$

(b) Use separation of variables to find a solution to (11) subject to the initial condition $N(0) = N_o$.

***Solution**:* Separate variables to obtain

$$\int \frac{1}{N}\, dN = \int k_o\, e^{-t}\, dt,$$

which integrates to

$$\ln |N| = -k_o e^{-t} + c_1, \tag{12}$$

for some constant c_1. Taking the exponential function on both sides of (12) yields

$$|N| = c_2 \exp(-k_o e^{-t}), \tag{13}$$

where we have set $c_2 = e^{c_1}$. Finally, using the continuity of the exponential function, we obtain from (**??**) that

$$N(t) = c\ \exp(-k_o e^{-t}), \qquad \text{for all } t \in \mathbf{R}, \tag{14}$$

for a constant c. Using the initial condition $N(0) = N_o$, we obtain from (**??**) that

$$c\ \exp(-k_o) = N_o,$$

which yields

$$c = N_o\ \exp(k_o). \tag{15}$$

Substituting the value of c in (15) into (**??**) yields

$$N(t) = N_o \ \exp(k_o(1 - e^{-t})), \quad \text{for all } t \in \mathbf{R}. \tag{16}$$

□

(c) What does the model predict about the number of individuals in the population in the long run.

Solution: It follows from (16) that

$$\lim_{t\to\infty} N(t) = N_o e^{k_o}; \tag{17}$$

so that the population size will tend towards the limiting value of $e^{k_o} N_o$. □

(d) **(Bonus)** Given that the population doubles after one unit of time, find k_o and compute

$$\lim_{t\to\infty} N(t).$$

Solution: Given that $N(1) = 2N_o$, it follows from (17) that

$$N_o \ \exp(k_o(1 - e^{-1})) = 2N_o,$$

which can be solved for k_o to yield

$$k_o = \frac{\ln 2}{1 - e^{-1}}. \tag{18}$$

Combining (18) and (17) yields

$$\lim_{t\to\infty} N(t) = N_o \ \exp(\ln 2/(1 - e^{-1})).$$

□