Review Problems for Exam 2

1. Suppose that the growth of a population of size $N = N(t)$ follows the differential equation model
$$\frac{dN}{dt} = aN - b, \tag{1}$$
where a and b are positive parameters.

   (a) Give an interpretation for the model in (1).

   (b) Describe all possible behaviors predicted by the model in (1).

2. Find the equilibrium solutions of the differential equation $\dfrac{dy}{dt} = y^2 - 36$, and determine their stability properties.

3. We have seen that the (continuous) logistic model $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$, where r and K are positive parameters, has an equilibrium point at $\overline{N} = K$.

   (a) Let $f(N) = rN\left(1 - \dfrac{N}{K}\right)$ and give the linear approximation to $f(N)$ for N close to K.

   (b) Let $u = N - K$ and consider the linear differential equation
$$\frac{du}{dt} = f'(K)u.$$
This is called the *linearization* of the equation
$$\frac{dN}{dt} = f(N)$$
around the equilibrium point $\overline{N} = K$.
Use separation of variables to solve this equation. What happens to $|u(t)|$ as $t \to \infty$, where u is any solution to the linearized equation?

   (c) Use your result in the previous part to give an explanation as to why any solution to the logistic equation that begins very close to K can be approximation by $K + u(t)$, where u is a solution to the linearized equation.

   (d) Suppose that $N = N(t)$ is a solution to the logistic equation that starts at N_o, where N_o is very close to K. Find an estimate of the time it takes for the distance $|N(t) - K|$ to decrease by a factor of e. This time is called the *recovery time.*

4. Consider the first–order ordinary differential equation $\dfrac{dy}{dt} = y^2 - 2y + 1$.

   (a) Determine equilibrium points and determine the nature of the stability of the equilibrium solutions by means of the principle of linearized stability

   (b) Use separation of variables to find the general solution to the equation.

   (c) Use your result from the previous part to determine the nature of the stability of the equilibrium points.

   (d) Find a solution to the IVP $\begin{cases} \dfrac{dy}{dt} &= y^2 - 2y + 1; \\ y(0) &= 2, \end{cases}$ and determine its maximal interval of existence.

5. Let $F(t) = \displaystyle\int_0^t \tau^2 e^{-\tau}\ d\tau$ for all $t \in \mathbf{R}$.

   (a) Use integration by parts to evaluate $F(t)$.

   (b) Sketch the graph of $y = F(t)$.

6. Let $g\colon \mathbf{R} \to \mathbf{R}$ be given by $g(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$

   (a) Use the first linear approximation to sin around $a = 0$, with the corresponding error term, to compute $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$, and conclude that the function g defined above is continuous.

   (b) Use the first order approximation to sin around $a = 0$ to find an approximation for g around $a = 0$. Estimate the error in the approximation.

   (c) Use the result in (b) above to approximate $\displaystyle\int_0^x \frac{\sin t}{t}\ dt$. How good is your approximation?

7. Solve the initial value problem

$$\frac{dy}{dt} = y + t^2, \qquad y(0) = 0,$$

   and compute $\displaystyle\lim_{t\to\infty} y(t)$.

8. Solve the initial value problem

$$\frac{dy}{dt} = e^t \sin t, \qquad y(0) = 0.$$

9. Consider the first order differential equation $\dfrac{dy}{dt} = y^3 - 4y$.

   (a) Find all equilibrium solutions of the equation and determine the nature of their stability.

   (b) Sketch a few of the possible solutions to the equation.

10. The law of mass action states that the rate of a chemical reaction is proportional to the concentrations of the reacting substances.

    Consider a chemical reaction, $A + B \to C$, in which two substances, A and B, react to produce a single substance, C. Assume that the reverse reaction does not have a considerable effect and therefore can be neglected. Let $y = y(t)$ denote the number of kilograms of the reaction product, C, after t minutes. Suppose that the original amount of the reacting substances are 80 kilograms and 60 kilograms. As a consequence of the law of mass action, we obtain that

$$\frac{dy}{dt} = k(80 - y)(60 - y) \quad \text{for some constant } k > 0.$$

    That is, the rate of production of C is proportional to the product of the remaining amounts of the reactants A and B.

    (a) Sketch some possible solutions to the equation.

    (b) Use separation of variables to solve the above differential equation assuming that $y = 0$ when $t = 0$.

    (c) In part (b), assume also that there are 20 kilograms of the reaction product 10 minutes after the onset of the reaction. How much reaction product is present 5 minutes later?