Solutions to Additional Review Problems

1. An initial population of 50,000 inhabits a microcosm with carrying capacity of 100,000. Suppose that, after five years, the population increases to 60,000. Determine the intrinsic growth rate of the population.

   Solution: We assume that the growth of the population is governed by the Logistic equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$
where $K = 10^5$ in this case.

   The solution to the differential equation in (1) subject to the initial condition $N(0) = N_o$ is given by
$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}}, \qquad \text{for } t \geqslant 0. \tag{2}$$

   For the situation at hand, $N_0 = 5 \times 10^4$.

   Given that $N(5) = 6 \times 10^4$, we would like to determine the value of the intrinsic growth rate, r. In order to do this, we solve (2) for r. Writing N for $N(t)$ in (2) and taking reciprocals on both sides of the equation we obtain
$$\frac{1}{N} = \frac{N_o + (K - N_o)e^{-rt}}{N_o K},$$
which can be re–written as
$$\frac{1}{N} = \frac{1}{K} + \left(\frac{1}{N_o} - \frac{1}{K}\right)e^{-rt}. \tag{3}$$

   The equation in (3) can now be solved for e^{-rt} to yield
$$e^{-rt} = \frac{\dfrac{1}{N} - \dfrac{1}{K}}{\dfrac{1}{N_o} - \dfrac{1}{K}},$$
or
$$e^{-rt} = \frac{N_o}{N} \cdot \frac{K - N}{K - N_o}. \tag{4}$$

   Taking reciprocals on both sides of (4) yields
$$e^{rt} = \frac{N(K - N_o)}{N_o(K - N)}. \tag{5}$$

Taking the natural logarithm on both sides of (5) and solving for r, we obtain

$$r = \frac{1}{t}\ln\left(\frac{N(K-N_o)}{N_o(K-N)}\right). \tag{6}$$

Substituting the values $N = 6\times 10^4$, $N_o = 5\times 10^4$, $K = 10^5$, and $t = 5$ into (6) yields

$$r = \frac{1}{5}\ln\left(\frac{3}{2}\right).$$

□

2. Hydrocoden bitartrate is prescription drug used as a cough suppressant and pain reliever. Assume the drug is eliminated from the body by a natural decay process with half–life of 3.8 hours. The usual dose is 10 mg every 6 hours.

   (a) Use a conservation principle to derive a differential equation satisfied by the amount $Q(t)$ of the drug in the patient after a dose.

   ***Solution**:* Model the patient's bloodstream as a compartment of fixed volume. Let $Q = Q(t)$ denote the amount of drug in the compartment at time t. Apply the conservation principle

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}, \tag{7}$$

   where

$$\text{Rate of } Q \text{ in } = 0, \tag{8}$$

$$\text{Rate of } Q \text{ out } = \lambda Q, \tag{9}$$

   with $\lambda > 0$ being a constant of proportionality.

   Combining the equations (7)–(9), we obtain the differential equation

$$\frac{dQ}{dt} = -\lambda Q. \tag{10}$$

□

   (b) Assume that the amount of the drug in the patient prior to the dose is Q_o and that the drug is absorbed immediately. Give a formula for computing $Q(t)$, where t measures the length of time after the dose.

   ***Solution**:* The solution to the differential equation in (10) subject to the initial condition $Q(0) = Q_o$ is

$$Q(t) = Q_o e^{-\lambda t}, \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

The rate constant, λ, is related to the half–life, τ_2, by means of the equation

$$\lambda = \frac{\ln 2}{\tau_2}, \tag{12}$$

where $\tau_2 = 3.8$ hours in this case. Combining (11) and (12) yields the formula

$$Q(t) = Q_o e^{-\frac{\ln 2}{\tau_2} t}, \quad \text{for all } t \in \mathbb{R},$$

or

$$Q(t) = \frac{Q_o}{2^{t/3.8}},$$

where t is measured in hours. □

3. Suppose that alcohol is introduced into a 2-liter beaker, which initially contains distilled water, at a rate of 0.1 liners per minute. Assume that the a well–mixed mixture is removed from the beaker at the same rate.

   (a) Derive a differential equation for the concentration of alcohol in percent volume at any time t.

   ***Solution**:* The beaker is a compartment of fixed volume $V = 2$ liters. Let $Q = Q(t)$ denote the volume of alcohol in the compartment at time t. Apply the conservation principle

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}. \tag{13}$$

   Let F denote the rate at which alcohol flows into the beaker; in this case, $F = 0.1$ liters per minute. Then,

$$\text{Rate of } Q \text{ in } = F. \tag{14}$$

   Setting

$$c(t) = \frac{Q(t)}{V}, \quad \text{for all } t, \tag{15}$$

   the concentration of alcohol in the beaker at time t, in percent volume, we have that

$$\text{Rate of } Q \text{ out } = c(t)F,$$

   or

$$\text{Rate of } Q \text{ out } = \frac{F}{V}Q, \tag{16}$$

by virtue of (15).

Combining the equations (13), (14) and (16), we obtain the differential equation

$$\frac{dQ}{dt} = F - \frac{F}{V}\, Q, \tag{17}$$

for the volume of alcohol in the beaker at time t.

Next, divide the equation in (17) by V, and use (15) to obtain the differential equation

$$\frac{dc}{dt} = \frac{F}{V} - \frac{F}{V}\, c,$$

or

$$\frac{dc}{dt} = -\frac{F}{V}\,(c-1)\,. \tag{18}$$

□

(b) How long will it take for the concentration of alcohol to reach 50%?

***Solution**:* In order to answer this question, we solve the differential equation in (18) subject to the initial condition $c(0) = 0$, since the beaker starts one with 2 liters of distilled water. The solution to this initial value problem is

$$c(t) = 1 - e^{-\frac{F}{V}\,t}, \tag{19}$$

where t is measured in minutes.

Next, we solve the equation

$$c(t) = 0.5,$$

where $c(t)$ is given by (19), to obtain

$$t = \frac{V}{F}\,\ln 2 \doteq 13.86 \text{ minutes.}$$

□

4. The rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time.

   (a) Write and solve a differential equation for the quantity, Q, of the drug in the blood at time, t, in hours.

Solution: Model the bloodstream as a compartment and let $Q(t)$ denote the amount of the drug in the compartment. Apply the conservation principle

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}, \tag{20}$$

where

$$\text{Rate of } Q \text{ in } = 0, \tag{21}$$

$$\text{Rate of } Q \text{ out } = kQ, \tag{22}$$

with $k > 0$ being a constant of proportionality.

Combining the equations (20)–(22), we obtain the differential equation

$$\frac{dQ}{dt} = -kQ. \tag{23}$$

□

(b) Assume that 30% is left in the blood after 4 hours. How much of the drug is left in the patient's body after 6 hours if the patient is given 100 mg initially?

Solution: The solution to the differential equation in (23) subject to the initial condition $Q(0) = Q_o$ is given by

$$Q(t) = Q_o e^{-kt}, \quad \text{for all } t \in \mathbb{R}. \tag{24}$$

Given that $Q(3) = 0.3Q_o$, we obtain from (24) that

$$Q_o e^{-3k} = 0.3Q_o,$$

which can be solved for k to yield

$$k = -\frac{1}{3}\ln(0.3) \doteq 0.40. \tag{25}$$

Using the estimate for k in (25) we obtain that

$$Q(6) = 100e^{-6k} \doteq 9;$$

so that there are about 9 mg of the drug left in the patient's body after 6 hours. □