Calculus II with Applications to the Life Sciences

Preliminary Lecture Notes

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Contents

Chapter 1

Preface

This set of notes has been developed in conjunction with the teaching of Math 31S (Calculus II with Applications to the Life Sciences) at Pomona College during the fall semester of 2011. The main goal of the course is to introduce and develop some of the topics in a second semester Calculus course in the context of problems arising in the life sciences. In particular, we will study how integral and differential calculus can be applied to solve problems that come up in population biology; some of those problems are concerned with the description of the evolution in time of the size of the population of a given species, as well as the interaction of several species living in a common environment. Analysis of this type of problems leads naturally to differential equations. These are mathematical expressions involving an unknown function (which one seeks to find) and its derivatives.

We will spend the first part of the course learning how to analyze the differential equations that come up in the study of the problems mentioned above. Some of the equations can be solved using integral calculus, but others cannot be solved easily, and so the best one can do is to use approximations, in particular, linear approximations, to analyze them. We will see that sometimes those approximate solutions to the equations actually tell us a lot about the system we are studying.

Chapter 2

Introduction to Modeling

In this Chapter we introduce a very important modeling principle that can be used to model situations in the biological or physical sciences in which a certain quantity (e.g., amount of a substance in a system, number of individuals of a given species in an ecosystem, etc.) varies with time. We shall refer to it in this course as a **conservation principle**. Simply stated, a conservation principle stipulates that the rate of change of a the amount of a substance in a system is accounted for by the how much of the substance goes into the system per unit time, and how much of it goes out of the system per unit time. In mathematical terms, if we let $Q = Q(t)$ denote the amount of the substance in the system at time t and assume that Q is modeled by a differentiable function of time, the conservation principle can be stated as the equation

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out}. \tag{2.1}$$

The expression in (2.1) is an example of a differential equation. It is an equation because there is an unknown function, $Q = Q(t)$, that we seek to find. The adjective "differential" refers to the fact that the derivative of Q is involved in the expression.

We will see in this course that large part of what we call mathematical modeling reduces to postulating what specific form the right–hand side of the equation in (2.1) will take. Determining the actual form of the right–hand side of (2.1) involves some understanding of the system we are studying as well as some assumptions that are made about how the system works. This process might require application of scientific principles that govern the system or some empirical information that have been obtained about the system previously. We will illustrate this process by presenting two examples: one–compartment dilution and the derivation of a some models of population growth. The mathematical problems that will be derive in the next two sections are the ones that we will be interested in solving throughout the course.

2.1 One–Compartment Dilution

Imagine that a certain compartment of fixed volume, V, contains a solvent and substance that dissolves in the solvent. Let $Q = Q(t)$ denote the amount of the substance that is being dissolved and which is present in the compartment at time t depicted in Figure 2.1.1.

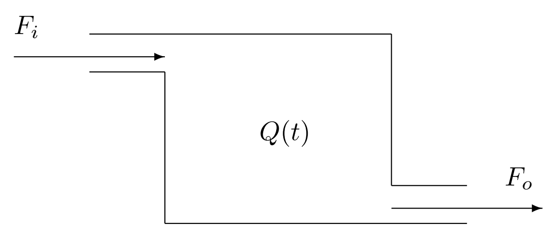

Figure 2.1.1: One–Compartment Model

We assume that solution at a concentration c_i of the substance goes into the compartment at a fixed rate F_i measured in units of volume per unit time (for example, milliliters per minute). Assume also that the solution in the compartment is mixed instantly. If we know the amount $Q(t)$, we can then determine the concentration of the substance in the compartment as a function of time by the expression

$$c(t) = \frac{Q(t)}{V}. \tag{2.2}$$

If the amount of the substance is measured in grams (gr), and the volume, V, is given in milliliters (ml), for instance, then $c(t)$ is measured in units of gr/ml.

We assume that $Q = Q(t)$ is a differentiable function of t and apply the conservation principle in (2.1) with

$$\text{Rate of } Q \text{ in} = c_i F_i \tag{2.3}$$

and

$$\text{Rate of } Q \text{ out} = c(t) F_o. \tag{2.4}$$

To see how (2.3) comes about, observe that in a small instance of time, Δt, a volume,

$$\Delta V = F_i \cdot \Delta t,$$

of solution at a concentration of c_i enters the compartment depicted in Figure 2.1.1. In that volume, ΔV, there is an amount of substance

$$\Delta Q_i = c_i F_i \cdot \Delta t. \tag{2.5}$$

Dividing the expression in (2.5) by $\Delta t \neq 0$, and letting Δt approach 0 yields the expression in (2.3) by virtue of the assumption of differentiability of Q. Similar considerations justify the expression in (2.4).

Combining the expressions in (2.1), (2.3), (2.4) and (2.5) yields the differential equation

$$\frac{dQ}{dt} = c_i F_i - \frac{F_o}{V}\, Q. \qquad (2.6)$$

In a practical setting, the parameters c_i, F_i, F_o and V are known, or can be estimated. So, what we would like to know is the quantity $Q = Q(t)$ at any time t. Usually, we have some information about $Q(t)$ at a specific time $t = t_o$; for instance, we may know the amount of the substance at the time that we begin to observe the system, or at some other time in the past. The task, then, is to determine the amount of the substance at all other times. In that case we will be using the model to make predictions which can be tested or which can be used make informed decisions about the system under study. We will see in subsequent sections how we can solve the differential equation in (2.6).

2.2 Modeling Population Growth

In this section we will see how to use a conservation principle like the one stated in equation (2.1) in order to derive a mathematical model that can be used to describe how the number of organisms of a given species living in certain environment varies with time. The simplest example in the context of this course is that of the number of bacteria in a given culture.

Let $N = N(t)$ denote the number of individuals of a population living in a certain region or environment. We would like to answer the following question: Suppose the size of the population, N_o, is known at a certain time t_o (that is, $N_o = N(t_o)$ is known), is it possible to determine $N(t)$ for other times t? In order to apply a conservation principle of the type given in (2.1) to the quantity $N(t)$, we need to assume that $N = N(t)$ is a differentiable function of time. This assumption is reasonable for situations in which

(i) we are dealing with populations of very large size so that the addition (or removal) of a few individuals is not very significant; for example, in the case of a bacterial colony, N is of the order of 10^6 cells per milliliter;

(ii) "there are no distinct population changes that occur at timed intervals," see [EK88, pg. 117].

Under these assumptions, we may invoke the following conservation principle:

$$\frac{dN}{dt} = \text{Rate of individuals in} - \text{Rate of individuals out}; \qquad (2.7)$$

that is, any change in the population size in a given region or environment has to be accounted for by the number of new individuals, per unit time, that are added to the population minus those that are taken out of the population. A more specific form the *conservation equation* (2.7) would be

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migrations} - \text{harvesting} + \text{etc.}, \qquad (2.8)$$

where all the quantities on the right–hand side of the equation in (2.8) are given per unit of time; in other words, they are given as *rates.* Rates in population studies are usually given *per capita;* that is, per unit of population. Thus, the conservation principle (2.8) can be further written as

$$\frac{1}{N}\frac{dN}{dt} = \text{birth rate} - \text{death rate} + \text{migration rate} + \text{etc.}, \tag{2.9}$$

where all the rates on the right–hand side are *per capita* rates.

For example, in the case in which there are no migrations in or out of the population, no harvesting or predation, etc., the model in (2.9) takes on the simpler form:

$$\frac{1}{N}\frac{dN}{dt} = \textit{per capita} \text{ birth rate} - \textit{per capita} \text{ death rate}. \tag{2.10}$$

The *per capita* on the right–hand side of (2.10) need to be modeled. In general, these can be assumed to be continuous functions of many variables; for instance, in order to take into account seasonal effects on the *per capita* birth and death rates, as well as effects due to overcrowding, or limited space, we may assume that that the rates are functions of t and N, the population size. More generally, we may write

$$\textit{per capita} \text{ birth rate} = b(t, N, \text{nutrients or resources, etc.})$$

and

$$\textit{per capita} \text{ death rate} = d(t, N, \text{nutrients or resources, etc.}),$$

where b and d are continuous functions of all the factors inside the parentheses. The equation in (2.10) then yields

$$\frac{dN}{dt} = (b - d)N, \tag{2.11}$$

where it is understood that the functions b and d may depend on several variables. Setting

$$a = b - d,$$

we may write the differential equation in (2.11)

$$\frac{dN}{dt} = aN, \tag{2.12}$$

where a is the *per capita,* or relative, growth rate of the population, which may depend on many factors.

Making assumptions on the type of dependence of the functions b and d on the various variables leads to different population models based on (2.11) or (2.11). In the remaining of this section we discuss two of those models.

First, assume that b and d are constant; that is, b and d are actually independent of t, N, etc., In this case , we get that the *per capita* growth rate, a, is a constant, denoted by a_o, so that the equation in (2.12) becomes

$$\frac{dN}{dt} = a_o N. \tag{2.13}$$

The equation in (2.13) is known as the Malthusian model; an analysis of this model, which will be presented later in the course, will show that (2.13) predicts unlimited, exponential, grown if $a_o > 0$, and exponential decay in $a_o < 0$.

A more realistic model for population growth is obtained by assuming that the functions b and d in (2.11) depend only on N, the population size, and are linear functions of N:

$$b = b_o - \alpha N, \tag{2.14}$$

and

$$d = d_o + \beta N, \tag{2.15}$$

for some positive constants, b_o, d_o, α and β, with $d_o < b_o$. The functional forms in (2.14) and (2.15) may be thought of as modeling the effects of overcrowding and competition for resources in the population. Substituting the expressions for b and d in (2.14) and (2.15) into (2.11) leads to the equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{2.16}$$

where $r > 0$ is called *intrinsic growth rate,* which approximates the *per capita* growth rate for very low population densities, and $K > 0$ is called the *carrying capacity.* The equation in (2.16) is known as the Logistic growth model of population growth. Analysis of this model will show that (2.16) predicts limited growth; that is, for situations in which the initial population, $N(0) = N_o$, is positive, equation (2.16) will have a solution, $N = N(t)$, satisfying

$$\lim_{t\to\infty} N(t) = K.$$

Our main goal in this course is to show how differential and integral Calculus can be used analyze differential equation models like the ones in (2.6), (2.12) and (2.16), and more general ones. We will also see how the theory of approximations in differential Calculus can be used to obtain qualitative information about what the models predict.

Chapter 3

Applications of Differential Calculus: Part I

In this chapter we show how to use the concepts and techniques of differential Calculus that students learned in Calculus I in order to give a preliminary analysis of the Logistic equation in (2.16). The analysis provided here assumes that a solution to (2.16) satisfying a given initial condition exists, and is unique. Existence and uniqueness for the initial value problem for (2.16) will be obtained in Chapter 4 as an application of integral Calculus.

3.1 Preliminary Analysis of the Logistic Equation

We assume that a solution, $N = N(t)$, to the Logistic equation in (2.16) exists for any non–negative initial condition

$$N(0) = N_o. \tag{3.1}$$

We also assume that the solution, $N = N(t)$, to (2.16) subject to the initial condition in (3.1) is twice differentiable.

Rewrite the differential equation in (2.16) in the form

$$\frac{dN}{dt} = \frac{r}{K} N\left(K - N\right), \tag{3.2}$$

and observe that, when $N = 0$ or $N = K$, $\dfrac{dN}{dt} = 0$. We will see in Chapter 4 that, in fact,

$$\frac{dN}{dt} = 0 \quad \text{for all } t,$$

so that $N(t)$ is constant (see Problem 1 in Assignment 1). Hence, we get that

$$N(t) = 0, \quad \text{for all } t, \tag{3.3}$$

and

$$N(t) = K, \qquad \text{for all } t, \tag{3.4}$$

are possible solutions to the Logistic equation in (3.2) corresponding to the initial conditions $N_o = 0$ and $N_o = K$, respectively. The solutions in (3.3) and (3.4) are known as equilibrium solutions of the Logistic equation in (3.2). They are sketched in Figure 3.1.1 in a graph of N versus t.

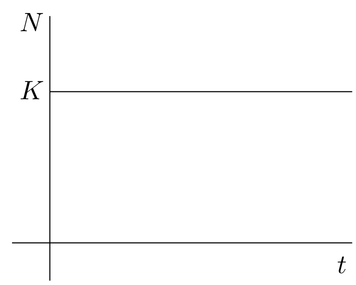

Figure 3.1.1: Sketch of Equilibrium Solutions

In the remainder of this section we will see how to use techniques of differential Calculus to obtain sketches of possible solutions of the initial value problem

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right); \\ \\ N(0) = N_o, \end{cases} \tag{3.5}$$

for $N_o > 0$ and $N_o \neq K$.

We see from (3.2) that $\dfrac{dN}{dt} > 0$ for $0 < N < K$; so that $N(t)$ increases for positive values of N less than K. Similarly, according to (3.2) again, $\dfrac{dN}{dt} < 0$ for $N > K$; it then follows from (3.2) that $N(t)$ decreases for values of N higher than K

Next, differentiate with respect to t on both sides of (3.2) to obtain

$$\frac{d^2N}{dt^2} = r\frac{dN}{dt} - \frac{2r}{K}\, N\frac{dN}{dt}, \tag{3.6}$$

where we have used the Chain Rule when taking the derivative of the second term on the right–hand side (3.6). The right–hand side of the equation in (3.6) can be factored to yield

$$\frac{d^2N}{dt^2} = \frac{2r}{K}\left(\frac{K}{2} - N\right)\frac{dN}{dt}. \tag{3.7}$$

Substituting the expression for $\frac{dN}{dt}$ in (3.2) into the right–hand side of the equation in (3.7) yields

$$\frac{d^2N}{dt^2} = \frac{2r^2}{K^2}\, N\left(N - \frac{K}{2}\right)(N - K)\,. \tag{3.8}$$

According to (3.8), the graph of $N = N(t)$ might have an inflection point at the values

$$N = 0, \quad N = \frac{K}{2}, \quad \text{or} \quad N = K.$$

We also get from (3.8) that the sign of the second derivative of N with respect to t, for positive values of N, is determined by the signs of the right–most factors on the right–hand side of (3.8):

$$N - \frac{K}{2} \quad \text{and} \quad N - K.$$

The signs of these two factors are displayed in Table 3.1. The concavity of of

	$0 < N < K/2$	$K/2 < N < K$	$N > K$
$N - \frac{K}{2}$	$-$	$+$	$+$
$N - K$	$-$	$-$	$+$
	0	$K/2$	K
$N''(t)$	$+$	$-$	$+$
graph of $N(t)$	concave–up	concave–down	concave–up

Table 3.1: Concavity of the graph of $N = N(t)$

the graph of $N = N(t)$ is also displayed in Table 3.1. From that table we get that the graph of $N = N(t)$ is concave up for

$$0 < N < \frac{K}{2} \quad \text{or} \quad N > K,$$

and concave down for

$$\frac{K}{2} < N < K.$$

Putting together the information on concavity in Table 3.1 and the fact that $N(t)$ increases for $0 < N < K$ and decreases for $N > K$, we obtain the sketches of possible solutions to the logistic equation displayed in Figure 3.1.2.

3.2 Mathematical Questions

The sketch of possible solutions to the Logistic equation (3.2) displayed in Figure 3.1.2 raises more questions than it actually answers. The sketches in the figure

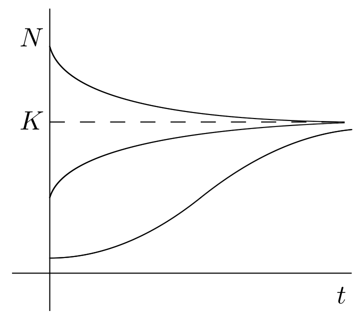

Figure 3.1.2: Possible Solutions to Logistic equation

presuppose certain properties of the solutions that have not been derived from the model yet. For instance, how do we know that graphs of different solutions do not cross, or intersect, each other? For example, how do we know that a population that starts out between $K/2$ and K, but very close to K, does not reach the carrying capacity in a finite time? How do we know that graphs of solutions can be drawn indefinitely as t increases? Do solutions exists for all positive values of t? These equations will be answered for the Logistic equation in the following chapter.

In Chapter 4 we will use integral Calculus to show that solutions to the initial value problem in (3.5) corresponding $N_o \geqslant 0$ exist for all $t \geqslant 0$. We will also see that graphs of solutions corresponding to different initial conditions cannot intersect. The first statement answers the mathematical question of global existence, while the second relates to the question of uniqueness.

Chapter 4

Applications of Integral Calculus

Differential equation models like the ones discussed in Chapter 2 seek to describe the behavior of a quantity, $y = y(t)$, as it varies with time, t. The conservation principle that was described is that chapter leads to differential equations of the general form

$$\frac{dy}{dt} = f(t, y, \lambda), \tag{4.1}$$

where lambda represents a parameter, or set of parameters. The equation in (4.1) prescribes the rate of change of the function $y = y(t)$ by some function, $f(t, y, \lambda)$, of the independent variable, t, the dependent variable y, and (possibly) some parameter, or set of parameters, λ, which are determined by the physical or biological situation being modeled. In the simplest situation, equation (4.1) takes the form

$$\frac{dy}{dt} = f(t), \tag{4.2}$$

for some continuous function, f, of a single variable, t. In this situation, solving the problem posed by the differential equation model reduces to answering the question:

Question 4.0.1. *Suppose that we know the rate of change of a function, $y = y(t)$, at every time, t, in some time–interval. Can we recover the function $y = y(t)$ from that information?*

We will see in the next section that Question 4.0.1 is answered by the Fundamental Theorem of Calculus, provided that we have information on the value of the function $y = y(t)$ at some initial time, t_o:

$$y(t_o) = y_o. \tag{4.3}$$

The equations in (4.2) and (4.3), taken together, are known as an initial value problem for the differential equation in (4.2) and usually written as

$$\begin{cases} \dfrac{dy}{dt} = f(t), \\ y(t_o) = y_o. \end{cases} \tag{4.4}$$

4.1 Recovering a Function from its Rate of Change

Writing the differential equation in the initial value problem (4.4) as

$$y'(\tau) = f(\tau), \quad \text{for } \tau \in I, \tag{4.5}$$

and integrating on both side of (4.5) from t_o to t, we obtain

$$\int_{t_o}^{t} y'(\tau)\, d\tau = \int_{t_o}^{t} f(\tau)\, d\tau, \quad \text{for } t \in I. \tag{4.6}$$

Next, assuming that f is continuous on I, and applying the Fundamental Theorem of Calculus to the left–hand side of the equation in (4.6), yields

$$y(t) - y(t_o) = \int_{t_o}^{t} f(\tau)\, d\tau, \quad \text{for } t \in I. \tag{4.7}$$

Thus, using the initial condition (4.3) and solving for $y(t)$ in (4.7) yields

$$y(t) = y_o + \int_{t_o}^{t} f(\tau)\, d\tau, \quad \text{for } t \in I. \tag{4.8}$$

The fact that the function $y = y(t)$, for $t \in I$, defined in (4.8) solves the initial value problem in (4.4), for the case in which f is continuous, follows from the Fundamental Theorem of Calculus and the fact that

$$\int_{t_o}^{t_o} f(\tau)\, d\tau = 0. \tag{4.9}$$

To see why the function $y = y(t)$ defined in (4.8) solves the initial value problem in (4.4) for a continuous function f, first note that (4.9) and (4.8) imply that

$$y(t_o) = y_o + \int_{t_o}^{t_o} f(\tau)\, d\tau = y_0,$$

which shows that $y = y(t)$ satisfies the initial condition in (4.4).

Next, put

$$G(t) = \int_{t_o}^{t} f(\tau)\, d\tau, \quad \text{for all } t \in I. \tag{4.10}$$

Then, since we are assuming that f is continuous on I, the Fundamental Theorem of Calculus implies that G is differentiable on I and

$$G'(t) = f(t), \quad \text{for all } t \in I; \tag{4.11}$$

in other words, the function G is an antiderivative of f (see [Sil89, p. 147]).

In view of (4.8) and (4.10), note that

$$y(t) = y_o + G(t), \quad \text{for } t \in I, \tag{4.12}$$

so that, differentiating on both sides of (4.12) with respect to t and observing that y_o is a constant, we obtain from (4.12) and (4.11) that

$$y'(t) = f(t), \quad \text{for all } t \in I,$$

which shows that the function $y = y(t)$ defined in (4.8), for a continuous function f, satisfies the differential equation in (4.4).

Next, we see that there is only one differentiable which solves the initial value problem in (4.4). We say that he initial value problem in (4.4) has a unique solution. To establish the uniqueness of the solution to the initial value problem (4.4), suppose that $y = y(t)$ and $z = z(t)$ are two solutions of the initial value problem in (4.4) over some interval I; that is, suppose that y and z are differentiable functions satisfying

$$y'(t) = f(t) \quad \text{and} \quad z'(t) = f(t), \quad \text{for } t \in I, \tag{4.13}$$

and

$$y(t_o) = z(t_o) = y_o. \tag{4.14}$$

Put $v(t) = y(t) - z(t)$ for all $t \in I$. Then, v is differentiable on I, and

$$v'(t) = y'(t) - z'(t) = f(t) - f(t) = 0, \quad \text{for all } t \in I,$$

where we have used (4.13). It then follows that

$$v(t) = c, \quad \text{for all } t \in I, \tag{4.15}$$

where c is a constant (see Problem 1 in Assignment 1).

Next, use (4.15) and (4.14) to compute

$$c = v(t_o) = y(t_o) - z(t_o) = y_o - y_o = 0.$$

Hence, it follows from (4.15) that

$$v(t) = 0, \quad \text{for all } t \in I,$$

which implies that

$$z(t) = y(t), \quad \text{for all } t \in I;$$

We have therefore shown that any two solutions of the differential equation in (4.2) that agree at a point in the interval I must agree at every point in the interval. This proves uniqueness of the solution to the initial value problem in (4.4).

Example 4.1.1. Find the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = 1 - t^2, \\ y(1) = 2. \end{cases} \tag{4.16}$$

Solution: Applying the formula in (4.8) we obtain that

$$\begin{aligned} y(t) &= 2 + \int_1^t (1 - \tau^2)\, d\tau \\ &= 2 + \left[\tau - \frac{\tau^3}{3}\right]_1^t \\ &= 2 + t - \frac{t^3}{3} - \frac{2}{3}, \end{aligned}$$

so that

$$y(t) = \frac{4}{3} + t - \frac{t^3}{3}, \qquad \text{for all } t \in \mathbb{R},$$

is the unique solution to the initial value problem in (4.16). □

Example 4.1.2. Let $\ell = \ell(t)$ for $t > 0$ denote the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{t}; \\ y(1) = 0, \end{cases} \tag{4.17}$$

for $t > 0$. Sketch the graph of $y = \ell(t)$.

Solution: From $\ell'(t) = \dfrac{1}{t}$, for $t > 0$, we see that $\ell'(t) > 0$ for all $t > 0$, and therefore $\ell(t)$ is increasing for $t > 0$. Differentiating one more time we obtain $\ell''(t) = -\dfrac{1}{t^2}$, for $t > 0$; thus, $\ell''(t) < 0$ for $t > 0$, so that the graph of $y = \ell(t)$ is concave down for $t > 0$. A preliminary graph of the graph of $y = \ell(t)$ is shown in Figure 4.1.1 on page 21. □

Example 4.1.3. Let $\ell = \ell(t)$ for $t > 0$ denote the unique solution to the initial value problem (4.17) in Example 4.1.3, and let a denote a positive real number. Define

$$g(t) = \ell(at) - \ell(a), \qquad \text{for } t > 0$$

Compute $g'(t)$ and show that

$$\ell(at) = \ell(a) + \ell(t), \qquad \text{for all } t > 0. \tag{4.18}$$

Solution: Compute

$$g'(t) = \frac{d}{dt}[\ell(at)],$$

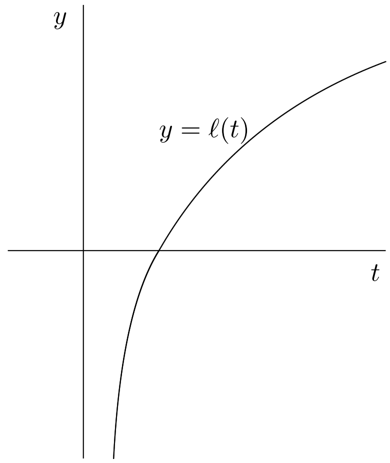

Figure 4.1.1: Sketch of graph of $y = \ell(t)$

since $\ell(a)$ is a constant. Next, apply the Chain Rule to obtain

$$g'(t) = \ell'(at)\frac{d}{dt}(at) = \frac{1}{at}\cdot a = \frac{1}{t}, \quad \text{for all } t > 0.$$

Thus, since $g(1) = \ell(a) - \ell(a) = 0$, it follows that $g(t)$ solves the initial value problem in (4.17). Consequently, since the the initial value problem in (4.17) has a unique solution, it follows that

$$g(t) = \ell(t), \quad \text{for all } t > 0;$$

in other words,

$$\ell(at) - \ell(a) = \ell(t), \quad \text{for all } t > 0.$$

We have therefore established the formula

$$\ell(at) = \ell(a) + \ell(t), \quad \text{for all } t > 0.$$

□

Example 4.1.4. Let $\ell = \ell(t)$ for $t > 0$ denote the unique solution to the initial value problem (4.17) in Example 4.1.3, and let p denote a non–zero real number. Define

$$h(t) = \frac{1}{p}\cdot \ell(t^p), \quad \text{for } t > 0.$$

Compute $h'(t)$ and show that

$$\ell(t^p) = p\,\ell(t), \quad \text{for } t > 0. \tag{4.19}$$

Solution: Apply the Chain Rule to compute

$$\begin{aligned} h'(t) &= \frac{1}{p}\cdot\frac{d}{dt}[\ell(t^p)] \\ &= \frac{1}{p}\cdot\ell'(t^p)\cdot\frac{d}{dt}[t^p] \\ &= \frac{1}{p}\cdot\frac{1}{t^p}\cdot p\cdot t^{p-1} \\ &= \frac{1}{t}, \end{aligned}$$

for all $t > 0$. Thus, since $h(1) = \frac{1}{p}\ell(1) = 0$, it follows that $h(t)$ solves the initial value problem in (4.17). Consequently, since the the initial value problem in (4.17) has a unique solution, it follows that

$$h(t) = \ell(t), \qquad \text{for all } t > 0,$$

or

$$\frac{1}{p}\cdot\ell(t^p) = \ell(t), \qquad \text{for all } t > 0,$$

from which we get

$$\ell(t^p) = p\ \ell(t), \qquad \text{for } t > 0.$$

□

4.2 The Natural Logarithm Function

The formulas in (4.18) and (4.19) in Examples 4.1.3 and 4.1.4, respectively, show that the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{t}; \\ y(1) = 0, \end{cases} \tag{4.20}$$

satisfies the properties of a logarithm. We will denote the solution to (4.20) by $\ln(t)$, for $t > 0$, and call it the natural logarithm of t for $t > 0$. The following properties of the natural logarithm function can be derived from its definition and the from work we did in Examples 4.1.3 and 4.1.4.

Proposition 4.2.1 (Properties of the Natural Logarithm)**.** Let $\ln(t)$ denote the unique solution to the initial value problem in (4.20). Then,

(i) $\ln(t) = \displaystyle\int_1^t \frac{1}{\tau}\,d\tau$, for $t > 0$;

(ii) $\ln(1) = 0$;

(iii) $\ln\colon (0,\infty) \to \mathbb{R}$ is differentiable and $\ln'(t) = \dfrac{1}{t}$, for all $t > 0$;

(iv) $\ln(ab) = \ln a + \ln b$ for all $a, b > 0$;

(v) $\ln(b^p) = p\ \ln b$ for all $b > 0$ and $p \in \mathbb{R}$,

Example 4.2.2. Define

$$g(u) = \ln(|u|), \qquad \text{for } u \neq 0.$$

Compute $g'(u)$ for $u \neq 0$.
Solution: Note that

$$|u| = \begin{cases} u & \text{if } u \geqslant 0; \\ -u & \text{if } u < 0. \end{cases} \tag{4.21}$$

We consider the cases $u > 0$ and $u < 0$ separately.

If $u > 0$, then

$$g(u) = \ln(u),$$

so that

$$g'(u) = \frac{1}{u}, \qquad \text{for } u > 0. \tag{4.22}$$

On the other hand, if $u < 0$, then, by the definition of $|u|$ in (4.21),

$$g(u) = \ln(-u),$$

so that, by the Chain Rule,

$$g'(u) = \ln'(-u) \cdot \frac{d}{du}(-u) = \frac{1}{-u} \cdot (-1) = \frac{1}{u};$$

thus,

$$g'(u) = \frac{1}{u}, \qquad \text{for } u < 0. \tag{4.23}$$

Combining the results of (4.22) and (4.23) we obtain

$$\frac{d}{du} \ln|u| = \frac{1}{u}, \qquad \text{for all } u \neq 0. \tag{4.24}$$

□

The differentiation formula (4.24) that was derived in Example 4.2.2 gives rise to the following, very useful, integration formula

$$\int \frac{1}{u}\ du = \ln|u| + c. \tag{4.25}$$

The integration formula in (4.25) complements the formula

$$\int u^p\ du = \frac{u^{p+1}}{p+1} + c, \qquad \text{for } p \neq -1, \tag{4.26}$$

by indicating what happens in the case $p = -1$, which is not covered by (4.26). The formulas in (4.25) and (4.26) are helpful when evaluating many integrals by means of a change of variables. We present here several examples on how to do that.

Example 4.2.3. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \tan(t); \\ y(0) = 0, \end{cases} \tag{4.27}$$

for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.
Solution: Compute

$$y(t) = \int_0^t \tan(\tau)\, d\tau = \int_0^t \frac{\sin(\tau)}{\cos(\tau)}\, d\tau$$

by making the change of variable $u = \cos(\tau)$; so that, $du = -\sin(\tau)\, d\tau$ and

$$\begin{aligned} y(t) &= -\int_1^{\cos t} \frac{1}{u}\, du \\ &= -\left[\ln|u|\right]_1^{\cos t}, \end{aligned}$$

where we have used the integration formula in (4.25); thus,

$$y(t) = -\ln|\cos t|,$$

Consequently, using property (v) in Proposition 4.2.1,

$$\begin{aligned} y(t) &= \ln|\cos t|^{-1} \\ &= \ln|(\cos t)^{-1}| \\ &= \ln|\sec t|, \end{aligned}$$

for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$. Hence, the solution to the initial value problem in (4.27) is the function $y\colon \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ given by

$$y(t) = \ln|\sec t|, \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

□

Example 4.2.4. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t}{1+t^2}; \\ y(1) = 3, \end{cases} \tag{4.28}$$

for $t \in \mathbb{R}$.
Solution: Compute

$$y(t) = 3 + \int_1^t \frac{\tau}{1+\tau^2}\, d\tau$$

by making the change of variable $u = 1 + \tau^2$; so that, $du = 2\tau\, d\tau$ and

$$\begin{aligned} y(t) &= 3 + \frac{1}{2}\int_2^{1+t^2} \frac{1}{u}\, du \\ &= 3 + \frac{1}{2}\left[\ln|u|\right]_2^{1+t^2}, \end{aligned}$$

where we have used the integration formula in (4.25); thus, the solution to the initial value problem in (4.28) is given by

$$y(t) = 3 + \frac{1}{2}\ln|1+t^2| - \frac{1}{2}\ln|2|, \quad \text{for all } t \in \mathbb{R},$$

which can be rewritten as

$$\begin{aligned} y(t) &= 3 + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 2 \\ &= 3 - \ln\sqrt{2} + \ln\sqrt{1+t^2} \end{aligned}$$

for $t \in \mathbb{R}$, by virtue of property (v) in Proposition 4.2.1. □

In the remainder of this section we study the asymptotic properties of the natural logarithm function; in particular, we show that

$$\lim_{t\to\infty} \ln(t) = +\infty \tag{4.29}$$

and

$$\lim_{t\to 0^+} \ln(t) = -\infty. \tag{4.30}$$

In order to establish (4.29), we first get an estimate for $\ln 2$. From the definition of ln, or Property (i) in Proposition 4.2.1, we see that

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt. \tag{4.31}$$

Thus, geometrically, $\ln 2$ is the area under the graph $y = \dfrac{1}{t}$ above the t–axis, and between the lines $t = 1$ and $t = 2$ (see Figure 4.2.2 on page 26). We could approximate $\ln 2$ by the area of the inscribed rectangles shown in Figure 4.2.3. The sum of the areas of the rectangles pictured in Figure 4.2.3 is an underestimate for $\ln 2$, so that

$$\ln 2 \quad > \quad \text{area of inscribed rectangles}, \tag{4.32}$$

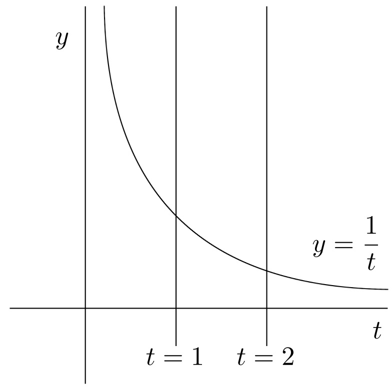

Figure 4.2.2: Sketch of graph of $y = 1/t$

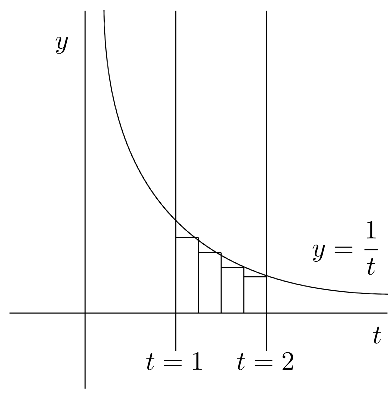

Figure 4.2.3: Inscribed Rectangles

where

$$\begin{aligned}\text{area of inscribed rectangles} &= \frac{1}{4}\left[\frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4} + \frac{1}{2}\right] \\ &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \qquad (4.33) \\ &= \frac{533}{840}.\end{aligned}$$

We therefore get from (4.32) and the result of the calculations in (4.33) that

$$\ln 2 > 0.63. \tag{4.34}$$

We are now ready to establish (4.29). For any $t > 2^n$, for some positive integer n, we have that

$$\ln t > \ln(2^n), \tag{4.35}$$

since ln is an increasing function. Next, apply Property (v) in Proposition 4.2.1 to get from (4.35) and (4.34) that

$$\ln t > (0.63)n. \tag{4.36}$$

Thus, we can make $\ln t$ as large as we please by making t large enough. For instance, say we want to make $\ln t$ larger than a million. Then, according to (4.36), we need to find a positive integer n such that

$$n > \frac{1,000,000}{0.63}; \tag{4.37}$$

for instance, we can take n to be 1.6×10^6, or 1.6 millions. Then, if $t > 2^{1600000}$, by virtue of (4.36) and (4.37), we are assured that $\ln t > 10^6$.

In general, given any big, positive, number M, we can find an integer, n, such that

$$n > \frac{M}{0.63}. \tag{4.38}$$

Then, if $t > 2^n$, it follows from the fact that ln is an increasing function that

$$\ln t > \ln(2^n) = n \ln 2 > (0.63)n, \tag{4.39}$$

where we have used the underestimate in (4.34). It then follows from (4.38) and (4.39) that

$$t > 2^n \text{ implies that } \ln t > M;$$

in other words, we can make $\ln t$ arbitrarily large and positive by making t sufficiently large. This is the meaning of the limit in (4.29).

In order to establish the limit in (4.30), we proceed in an analogous manner: We can make

$$\ln t < -M,$$

where M is an arbitrary large and positive number, by making

$$0 < t < \frac{1}{2^n}, \tag{4.40}$$

where n is a positive integer such that

$$n > \frac{M}{0.63},$$

the same condition in (4.38). In fact, if (4.40) holds true, then

$$0 < t < 2^{-n},$$

from which we get the

$$\ln t < \ln(2^{-n}), \tag{4.41}$$

since ln is an increasing function; consequently, using Property (v) in Proposition 4.2.1, we get from (4.41) that

$$\ln t < -n \ln 2 < -n(0.63), \tag{4.42}$$

where we have used the underestimate for $\ln 2$ in (4.34). Combining the estimate in (4.42) with (4.38) yields that, for any positive number M, if n is a positive integer chosen so that $n > M/(0.63)$, the

$$0 < t < 2^{-n} \text{ implies that } \ln t < -M,$$

which is equivalent to the statement in (4.30).

The limit expressions in (4.29) and (4.30), together with the continuity of the natural logarithm function, imply that the function $\ln\colon (0, \infty) \to \mathbb{R}$ maps positive real numbers onto $\mathbb{R}$; in other words, to every real number, y, there corresponds a positive number t such that

$$\ln t = y. \tag{4.43}$$

In addition, since $\ln(t)$ is strictly increasing, ln is a one–to–one function; that is, to every real number, y, there is exactly one solution, t to the equation in (4.43).

Definition 4.2.5 (The Number e)**.** We denote the unique solution to the equation

$$\ln t = 1$$

by the symbol e; thus, e is the unique real number with the property that

$$\int_1^e \frac{1}{\tau}\, d\tau = 1,$$

or

$$\ln e = 1. \tag{4.44}$$

Example 4.2.6. Find the unique solution to the equation

$$\ln x = -1. \tag{4.45}$$

Solution: Let $x = \dfrac{1}{e} = e^{-1}$. Then, using Property (v) in Proposition 4.2.1 and (4.44),

$$\ln(x) = \ln(e^{-1}) = (-1)\ln e = -1,$$

which shows that $x = e^{-1}$ is the unique solution to (4.45). □

4.3 The Number e

The number e is the unique real number with the property that

$$\ln e = 1. \tag{4.46}$$

In this section we obtain some estimates for e. We begin by showing that

$$2 < e < 3. \tag{4.47}$$

In order to establish (4.47), we first obtain an upper estimate for $\ln 2$ using the area of the circumscribed rectangles pictured in Figure 4.3.4. The area of the

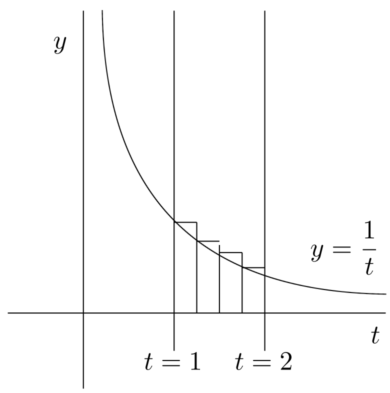

Figure 4.3.4: Circumscribed Rectangles

circumscribed rectangles in Figure 4.3.4 is an overestimate for $\ln 2$. We therefore have that

$$\ln 2 \quad < \quad \text{area of circumscribed rectangles}, \tag{4.48}$$

where

$$\begin{aligned} \text{area of circumscribed rectangles} \quad &= \quad \frac{1}{4}\left[1 + \frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4}\right] \\ &= \quad \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \\ &= \quad \frac{319}{420}. \end{aligned} \tag{4.49}$$

It follows from (4.48) and the calculations in (4.49) that

$$\ln 2 < 1,$$

so that

$$\ln 2 < \ln e, \tag{4.50}$$

where we have used (4.46). Next, use the fact that $\ln(t)$ is an increasing function of t to conclude from (4.50) that

$$2 < e. \tag{4.51}$$

To obtain the estimate on the right of (4.51), estimate $\ln 3$ by means of the area of the inscribed rectangles shown in Figure 4.3.5. The area of the inscribed

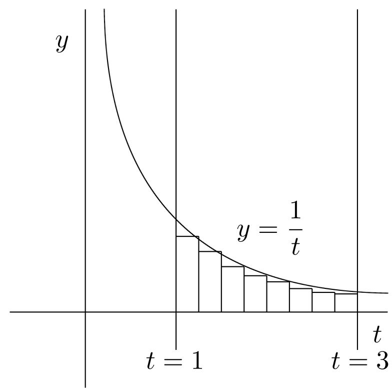

Figure 4.3.5: Inscribed Rectangles

rectangles shown in Figure 4.3.5 is an underestimate for $\ln 3$ so that

$$\text{area of inscribed rectangles} < \ln 3, \tag{4.52}$$

where

$$\begin{aligned} \text{area of inscribed rectangles} &= \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{12} \\ &= \frac{28271}{27720}, \end{aligned}$$

so that

$$\text{area of inscribed rectangles} \quad > \quad 1. \tag{4.53}$$

It follows from (4.52) and (4.53) that

$$1 < \ln 3,$$

so that

$$\ln e < \ln 3, \tag{4.54}$$

where we have used (4.46). Thus, using the fact that $\ln(t)$ is a strictly increasing function of t to conclude from (4.54) that

$$e < 3. \tag{4.55}$$

Combining the estimates in (4.51) and (4.55) yields (4.47).

In a similar manner, we can show that

$$2.5 < e < 2.875;$$

(see problems 1 and 2 in Assignment 6).

The number e is an irrational number that can be obtained as a limit of rational numbers by means of the formula

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{4.56}$$

Table 4.1 shows approximate values of the expression $\left(1+\frac{1}{n}\right)^n$ for n ranging over the first seven powers of 10, where the values in the second column are rounded up to five decimal places. We will establish formula (4.56) in Section

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827
1000000	2.71828
10000000	2.71828

Table 4.1: The sequence $\left(1+\frac{1}{n}\right)^n$ for $n = 10^k, k = 0, 1, \ldots, 7$

4.4 of these notes after we develop some properties of the exponential function. In the meantime, Table 4.1 shows that the five-decimal rational approximations of the sequence of real numbers,

$$a_n = \left(1+\frac{1}{n}\right)^n, \quad \text{for } n = 1, 2, 3, \ldots, \tag{4.57}$$

stabilize around 2.71828 for values of n equal to or higher than 10^6. We can use the value of 2.71828 as a five-decimal rational approximation to e; we write

$$e \doteq 2.71828, \tag{4.58}$$

where the dot above the equal sign indicates that the right–hand side of (4.58) is a rational approximation to e which is accurate to five decimal places.

The expression

$$e = \sum_{k=1}^{\infty} \frac{1}{k!}, \tag{4.59}$$

where

$$k! = 1 \cdot 2 \cdots (k-1) \cdot k$$

is the factorial of k, and the infinite sum on the right–hand side of (4.59) is understood as the limit of the sequence,

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!}, \tag{4.60}$$

of partial sums of the infinite sum on the right–hand side of (4.59); that is,

$$e = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!}. \tag{4.61}$$

Table 4.2 shows approximate values of the partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ rounded up to five decimal places. Note that we only need to go up to the 8^{th} term in the sequence of partial sums, s_n, in (4.60) to get the five–decimal approximation to e that we got using the sequence, a_n, in (4.57) by going to at least the one millionth term in the sequence.

n	$\sum_{k=0}^{n} \frac{1}{k!}$
1	2.00000
2	2.50000
3	2.66667
4	2.70833
5	2.71667
6	2.71806
7	2.71825
8	2.71828

Table 4.2: The sequence $\sum_{k=0}^{n} \frac{1}{k!}$ for $n = 0, 1, \ldots, 8$

4.4 The Exponential Function

We saw in Section 4.2 that the natural logarithm function, $\ln\colon (0,\infty) \to \mathbb{R}$ is a one–to–one function which maps the set of positive real numbers onto the set of all real numbers. It therefore has a inverse function that we denote by $\exp\colon \mathbb{R} \to (0,\infty)$ and call it the exponential function.

Definition 4.4.1 (Definition of Exponential Function)**.** We say that a positive real number y is the exponential of $t \in \mathbb{R}$, denoted $y = \exp(t)$, if and only if

$$\ln y = t.$$

In this section we derive the main properties of the exponential function. We will first obtain the graph of $y = \exp(t)$ by reflecting that of $y = \ln t$ in Figure 4.1.1 on the line $y = t$. A preliminary graph of $y = \exp(t)$ is shown in Figure 4.4.6. We see from the graph of the exponential function in Figure 4.4.6 that

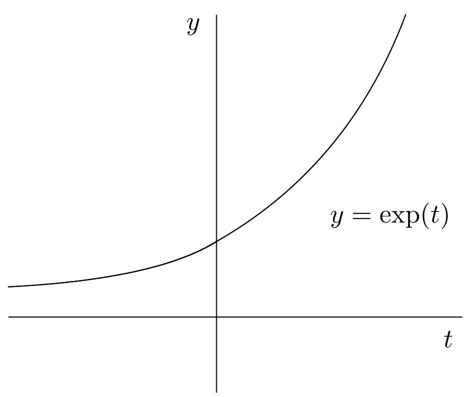

Figure 4.4.6: Sketch of graph of $y = \exp(t)$

$$\exp(0) = 1,$$

the graph of $y = \exp(t)$ is always concave up, and $\exp(t)$ is a strictly increasing function of t for all $t \in \mathbb{R}$.

It follows from the limits in (4.29) and (4.30) and the definition of the exponential function in Definition 4.4.1 that

$$\lim_{t\to+\infty} \exp(t) = +\infty \tag{4.62}$$

and

$$\lim_{t\to-\infty} \exp(t) = 0, \tag{4.63}$$

(see also the sketch of the graph of $y = \exp(t)$ in Figure 4.4.6).

Next, use the definition of the number e given in Definition 4.2.5 and Definition 4.4.1 to conclude that

$$\exp(1) = e. \tag{4.64}$$

We now turn to the differentiability properties of the exponential function. Since exp is the inverse of the natural logarithm function, which is a differentiable function whose derivative is never zero, it follows that exp is differentiable. Furthermore, applying the Chain Rule to the expression

$$\ln(\exp(t)) = t, \quad \text{for all } t \in \mathbb{R},$$

it follows that

$$\ln'(\exp(t)) \cdot \exp'(t) = 1, \quad \text{for all } t \in \mathbb{R},$$

from which we get

$$\frac{1}{\exp(t)} \cdot \exp'(t) = 1, \quad \text{for all } t \in \mathbb{R},$$

so that

$$\exp'(t) = \exp(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.65}$$

In other words, the function $y(t) = \exp(t)$ is a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y; \\ y(0) = 1, \end{cases} \tag{4.66}$$

for $t \in \mathbb{R}$. We will presently show that $\exp(t)$ is the only solution to the initial value problem in (4.66). Indeed, suppose that $v = v(t)$ is another solution to the initial value problem in (4.66); in other words, suppose that $v\colon \mathbb{R} \to \mathbb{R}$ is differentiable with

$$v'(t) = v(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.67}$$

and

$$v(0) = 1. \tag{4.68}$$

Then, consider the function

$$w(t) = \frac{v(t)}{\exp(t)}, \quad \text{for all } t \in \mathbb{R}, \tag{4.69}$$

which is well–defined since $\exp(t) > 0$ for all $t \in \mathbb{R}$. Furthermore, w is differentiable and, by the quotient rule,

$$w'(t) = \frac{\exp(t)v'(t) - v(t)\exp'(t)}{[\exp(t)]^2}, \quad \text{for all } t \in \mathbb{R}. \tag{4.70}$$

Next, use (4.65) and (4.67) to obtain from (4.70) that

$$w'(t) = \frac{\exp(t)v(t) - v(t)\exp(t)}{[\exp(t)]^2} = 0, \quad \text{for all } t \in \mathbb{R}.$$

Consequently,

$$w(t) = c, \text{ for all } t \in \mathbb{R}, \tag{4.71}$$

and some constant c by the result of Problem 1 in Assignment 1. Using the initial condition in (4.68) we get from (4.69) and (4.71) that

$$c = w(0) = \frac{v(0)}{\exp(0)} = 1;$$

so that, substituting the value of $c = 1$ on the right–hand side of (4.71), we get from (4.69) that

$$v(t) = \exp(t), \quad \text{for all } t \in \mathbb{R}.$$

We have therefore established that the initial value problem in (4.66) has the unique solution $y(t) = \exp(t)$ for all $t \in \mathbb{R}$. We will now derive a few consequences of this fact.

Example 4.4.2. Fix a real number a and define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \frac{1}{\exp(a)} \cdot \exp(a + t), \quad \text{for all } t \in \mathbb{R}. \tag{4.72}$$

Applying the Chain Rule we obtain that

$$\begin{aligned} f'(t) &= \frac{1}{\exp(a)} \cdot \exp'(a+t) \cdot \frac{d}{dt}(a+t) \\ &= \frac{1}{\exp(a)} \cdot \exp(a+t) \\ &= f(t), \end{aligned} \tag{4.73}$$

for all $t \in \mathbb{R}$, where we have used (4.65) and (4.72).

It follows from the calculations in (4.73) that f solves the differential equation in (4.66). Furthermore, using (4.72), we see that

$$f(0) = \frac{1}{\exp(a)} \cdot \exp(a) = 1,$$

so that f solves the initial value problem in (4.66). Since we have seen that problem (4.66) has a unique solution, namely the exponential function exp, it follows that

$$f(t) = \exp(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.74}$$

Using (4.73) and (4.74) we then have that

$$\exp(a + t) = \exp(a) \cdot \exp(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.75}$$

Example 4.4.3. Let r and y_o denote a real numbers and put

$$g(t) = y_o \exp(rt), \quad \text{for all } t \in \mathbb{R}. \tag{4.76}$$

Applying the Chain Rule we obtain that

$$\begin{aligned} g'(t) &= y_o \exp'(rt) \cdot \frac{d}{dt}[rt] \\ &= r[y_o \exp(rt)], \end{aligned} \tag{4.77}$$

where we have used (4.65). Combining the results of the calculations in (4.77) with (4.76) we see that

$$g'(t) = rg(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.78}$$

Furthermore, we obtain from (4.76) that

$$g(0) = y_o. \tag{4.79}$$

Thus, it follows from (4.78) and (4.79) that

$$y(t) = y_o \exp(rt), \quad \text{for all } t \in \mathbb{R}, \tag{4.80}$$

is a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ry; \\ y(0) = y_o, \end{cases} \tag{4.81}$$

where r and y_o are real constants. It can be shown that the function in (4.80) is the only solution to the initial value problem in (4.81); see Problem 2 in Assignment 7.

Example 4.4.4. In this example we derive the relation

$$[\exp(t)]^r = \exp(rt), \quad \text{for all } t \in \mathbb{R}, \tag{4.82}$$

and all $r \in \mathbb{R}$.

Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = [\exp(t)]^r, \quad \text{for all } t \in \mathbb{R}, \tag{4.83}$$

and note that

$$h(0) = 1. \tag{4.84}$$

Differentiate h in (4.83) with respect to t to obtain

$$\begin{aligned} h'(t) &= r[\exp(t)]^{r-1} \exp'(t) \\ &= r[\exp(t)]^{r-1} \exp(t) \\ &= r[\exp(t)]^r, \end{aligned} \tag{4.85}$$

where we have used the Chain Rule and (4.65). It follows from the calculations in (4.85) and the definition of h in (4.83) that

$$h'(t) = rh(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.86}$$

In view of (4.86) and (4.84), we see that h solves the initial value problem (4.81), with $y_o = 1$, given at the end of Example 4.4.3. Since $\exp(rt)$ is the unique solution to (4.81) (see Example 4.4.3), it follows from (4.83) and Problem 2 in Assignment 7 that

$$h(t) = \exp(rt), \quad \text{for all } t \in \mathbb{R}, \tag{4.87}$$

which establishes (4.82) by comparing (4.87) and (4.83).

We summarize the properties of the exponential function that we have derived so far in the following proposition.

Proposition 4.4.5 (Properties of the Exponential Function)**.** Let $\exp(t)$ denote the unique solution to the initial value problem in (4.66). Then,

(i) $\exp(0) = 1$;

(ii) $\exp(1) = e$;

(iii) $\exp\colon \mathbb{R} \to (0, \infty)$ is differentiable and $\exp'(t) = \exp(t)$, for all $t \in \mathbb{R}$;

(iv) $\exp(a + b) = \exp(a) \cdot \exp(b)$ for all $a, b \in \mathbb{R}$;

(v) $\exp(ab) = [\exp(a)]^b$ for all $a, b \in \mathbb{R}$.

Example 4.4.6. Applying properties (v) and (ii) in Proposition 4.4.5 we obtain for the case $e = 1$ and $b = t$, for any real number t, that

$$\exp(t) = [\exp(1)]^t = e^t, \quad \text{for all } t \in \mathbb{R}.$$

In view of the result of Example 4.4.6, we will adopt the following notation for the exponential function, $\exp\colon \mathbb{R} \to (0, \infty)$,

$$\exp(t) = e^t, \quad \text{for all } t \in \mathbb{R}. \tag{4.88}$$

The equation in (4.88) gives meaning to raising the real number e to the power t for any $t \in \mathbb{R}$. In fact, we can use the exponential and natural logarithm functions to give meaning to the expression b^a, where b is any positive real number and a is any real number as follows:

$$b^a = [\exp(\ln b)]^a = \exp(a \ln b). \tag{4.89}$$

Using the new notation for the exponential function introduced in (4.88), we can rephrase the properties stated in Proposition 4.4.5 as follows:

(i) $e^0 = 1$;

(ii) $e^1 = e$;

(iii) $\dfrac{d}{dt}[e^t] = e^t$, for all $t \in \mathbb{R}$;

(iv) $e^{a+b} = e^a \cdot e^b$ for all $a, b \in \mathbb{R}$;

(v) $e^{ab} = [e^a]^b$ for all $a, b \in \mathbb{R}$.

From the differentiation formula in (iii) we obtain integration formula

$$\int e^u \, du = e^u + c. \tag{4.90}$$

Example 4.4.7. Define $f(t) = te^{-t}$ for all $t \in \mathbb{R}$. We sketch the graph of $y = f(t)$.

First, we compute $f'(t)$ by applying the product rule and the Chain Rule to get

$$f'(t) = e^{-t} - te^{-t}, \qquad \text{for all } t \in \mathbb{R},$$

or

$$f'(t) = (1-t)e^{-t}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.91}$$

Since $e^{-t} > 0$ for all t, it follows from (4.91) that $f'(t) > 0$ for $t < 1$, and $f'(t) < 0$ for $t > 1$. Thus, $f(t)$ increases for $t < 1$ and decreases for $t > 1$. Thus, f has a local maximum at $t = 1$. The value of the local maximum is $f(1) = 1/e$.

Next differentiate $f'(t)$ in (4.91) with respect to t to obtain from (4.91) that

$$f''(t) = (t-2)e^{-t}, \qquad \text{for all } t \in \mathbb{R}, \tag{4.92}$$

where we have applied the product rule and the Chain Rule. We then obtain from (4.92) that $f''(t) < 0$ for $t < 2$, and $f''(t) > 0$ for $t > 2$, so that the graph of $y = f(t)$ is concave down for $t < 2$ and concave up for $t > 0$. The graph of $y = f(t)$ then has an inflection point at $(2, 2/e^2)$. A sketch of the graph of $y = te^{-t}$, for $t \in \mathbb{R}$, is shown in Figure 4.4.7. The sketch in Figure 4.4.7

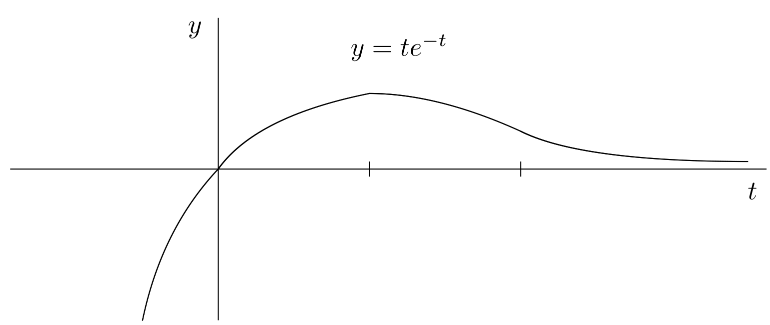

Figure 4.4.7: Sketch of graph of $y = te^{-t}$

incorporates information gained from the fact that

$$\lim_{t\to\infty} te^{-t} = 0,$$

which can be obtained by applying L'Hospital's Rule:

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

Example 4.4.8. Find the area under the graph of $y = e^{-t}$, for $t \in \mathbb{R}$, bounded by the coordinates axes and the line $t = 1$.
Solution: The region under consideration is sketched in Figure 4.4.6. The area

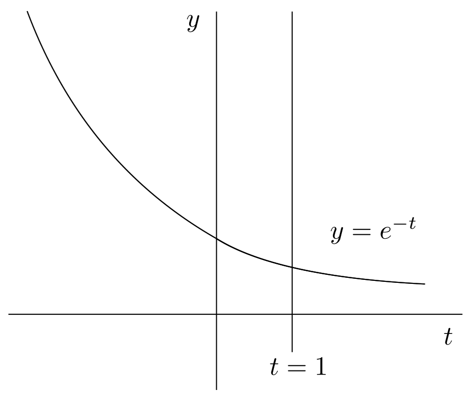

Figure 4.4.8: Sketch of graph of $y = e^{-t}$

of the region is given by

$$A = \int_0^1 e^{-t}\, dt. \tag{4.93}$$

Making the change of variables $u = -t$, so that $du = -dt$, we obtain from (4.93) that

$$\begin{aligned} A &= -\int_0^{-1} e^u\, du \\ &= \int_{-1}^{0} e^u\, du; \end{aligned} \tag{4.94}$$

Thus, using the integration formula in (4.90), we obtain from (4.94) that

$$A = [e^u]_{-1}^{0} = 1 - e^{-1}.$$

□

4.5 The Number e Revisited

In this section we use the properties of the natural logarithm and exponential functions that we have developed in the previous sections to establish (4.56);

namely,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{4.95}$$

Making the change of variables $h = 1/n$, so that $h \to 0$ as $n \to \infty$, we see that (4.95) is equivalent to

$$\lim_{h\to 0}(1+h)^{1/h} = e. \tag{4.96}$$

In the process of deriving (4.96) we will develop a method that is useful in computing other limits similar to those in (4.95) and (4.96).

We begin by defining the function setting

$$f(h) = (1+h)^{1/h}, \tag{4.97}$$

so that

$$\ln[f(h)] = \frac{1}{h}\ln(1+h), \quad \text{for } h > -1, \text{ and } h \neq 0, \tag{4.98}$$

where we have used property (v) in Proposition 4.2.1. Observe that, by property (i) in Proposition 4.2.1, we can rewrite (4.98) as

$$\ln[f(h)] = \frac{\ln(1+h)-\ln(1)}{h}, \quad \text{for } h > -1, \text{ and } h \neq 0. \tag{4.99}$$

It follows from (4.99) and the definition of the derivative of $\ln t$ at $t = 1$ that

$$\lim_{h\to 0}\ln[f(h)] = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \ln'(1) = 1. \tag{4.100}$$

Now, it follows from (4.98) and the definition of the exponential function that

$$f(h) = \exp(\ln[f(h)]), \quad \text{for } h > -1, \text{ and } h \neq 0. \tag{4.101}$$

Next, use the continuity of the exponential function and (4.101) to conclude from (4.100) that the $\lim\limits_{h\to 0} f(h)$ exists and equals

$$\lim_{h\to 0} f(h) = \exp\left(\lim_{h\to 0}\ln[f(h)]\right) = \exp(1) = e, \tag{4.102}$$

where we have used property (ii) in Proposition 4.4.5. Combining (4.102) and (4.97), we obtain (4.96), which was to be shown.

Alternatively, we could have computed the limit in (4.100) by applying L'Hospital's rule; in fact, since

$$\lim_{h\to 0}\ln(1+h) = \ln(1) = 0 \quad \text{and} \quad \lim_{h\to 0} h = 0,$$

L'Hospital's rule does apply to this situation and

$$\begin{aligned}
\lim_{h\to 0}\ln[f(h)] &= \lim_{h\to 0}\frac{\ln(1+h)}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{1}{1+h}}{1} \\
&= 1.
\end{aligned}$$

The procedure illustrated here can be used to compute limits of expressions in which the variable appears in the exponent. First, compute the natural logarithm of the expression. Then, compute the limit of the natural logarithm of the expression. If the limit of the natural logarithm of the expression exists, then the limit of the original expression exists and equals the exponential of the limit of the logarithmic expression.

Example 4.5.1. Let r denote a real number. Compute $\lim_{h\to 0}(1+rh)^{1/h}$.

Solution: Put $g(h) = (1+rh)^{1/h}$ so that

$$\ln[g(h)] = \frac{\ln(1+rh)}{h}.$$

Next, use L'Hospital's Rule to compute

$$\begin{aligned} \lim_{h\to 0}\ln[g(h)] &= \lim_{h\to 0}\frac{\ln(1+rh)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{1}{1+rh}\cdot\dfrac{d}{dh}(1+rh)}{1} \\ &= \lim_{h\to 0}\frac{\dfrac{r}{1+rh}}{1} \\ &= r. \end{aligned} \tag{4.103}$$

Consequently,

$$\begin{aligned} \lim_{h\to 0} g(h) &= \lim_{h\to 0}\exp(\ln[g(h)]) \\ &= \exp(\lim_{h\to 0}\ln[g(h)]) \\ &= \exp(r), \end{aligned}$$

where we have used the continuity of the exponential function and the results of the calculations in (4.103). We have therefore proved that

$$\lim_{h\to 0}(1+rh)^{1/h} = e^r. \tag{4.104}$$

□

By making the change of variables $n = 1/h$, so that $n \to \infty$ as $h \to 0$, we obtain from (4.104) that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r. \tag{4.105}$$

4.6 Analysis of the Malthusian Model

In this section we solve the Malthusian model (2.12),

$$\frac{dN}{dt} = aN, \tag{4.106}$$

where

$$a = b - d, \tag{4.107}$$

the difference of the *per-capita* birth and death rates, for the case in which b and d are constant. We will solve (4.106) subject to the initial condition

$$N(t_o) = N_o; \tag{4.108}$$

that is, the population size is known at some initial time t_o. We are thus lead to the study of the initial value problem

$$\begin{cases} \dfrac{dN}{dt} = aN; \\ N(t_o) = N_o, \end{cases} \tag{4.109}$$

where a is a real constant.

The initial value problem in (4.109) has the unique solution given by

$$N(t) = N_o e^{a(t-t_o)}, \quad \text{for } t \in \mathbb{R}, \tag{4.110}$$

(see Problem 5 in Assignment 8).

We study the behavior of the solution in (4.110) for the case $t_o = 0$, $N_o > 0$ and various values of (constant) *per–capita* growth rate a.

First, assume that $a > 0$; according to (4.107), this case correspond to a situation in which the (constant) *per–capita* birth rate, b, is higher than the *per–capita* death rate, d. In this case the Malthusian model in (4.106) predicts unlimited, exponential growth as pictured in Figure 4.6.9. If $a = 0$, the model in (4.106) predicts that the population size will remain constant, as shown in Figure 4.6.10. This corresponds to the situation in which the birth and death rates are exactly the same and cancel each other one generating a situation that can be said to be at equilibrium.

Finally, if $a < 0$, the model (4.106) predicts the situation sketched in Figure 4.6.11, which corresponds to exponential decay. In this case, the Malthusian model predicts that the population will go extinct.

Example 4.6.1 (Doubling Time)**.** Suppose that $a > 0$ for the Malthusian model in (4.109), where $t_o = 0$. Then, according to (4.110), the population size at any time $t \in \mathbb{R}$ is given by

$$N(t) = N_o e^{at}, \quad \text{for } t \in \mathbb{R}. \tag{4.111}$$

We can use (4.111) to find the time, $t = \tau_2$, at which the population size is double that of the initial population; in other words,

$$N(\tau_2) = 2N_o. \tag{4.112}$$

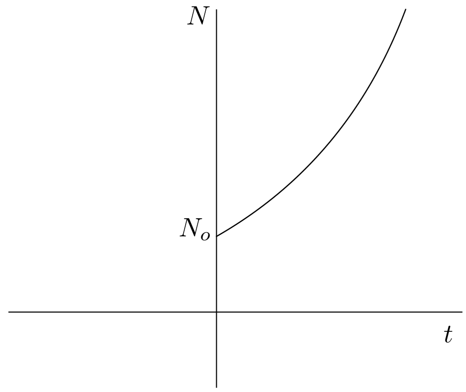

Figure 4.6.9: Sketch of graph of $N = N(t)$ for $a > 0$

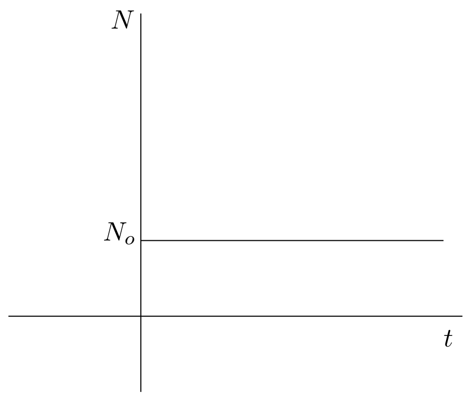

Figure 4.6.10: Sketch of graph of $N = N(t)$ for $a = 0$

Substituting τ_2 for t in (4.111) and using (4.112) we can write the equation

$$N_o e^{a\tau_2} = 2N_o. \tag{4.113}$$

Canceling N_o from both sides of (4.113) leads to the equation

$$e^{a\tau_2} = 2, \tag{4.114}$$

which can be solved for τ_2 by taking the natural logarithm function on both sides of (4.114) to yield

$$\tau_2 = \frac{\ln 2}{a}. \tag{4.115}$$

The expression in (4.115) defines the doubling time of a population undergoing Malthusian growth at a constant *per–capita* growth rate $a > 0$. The equation in

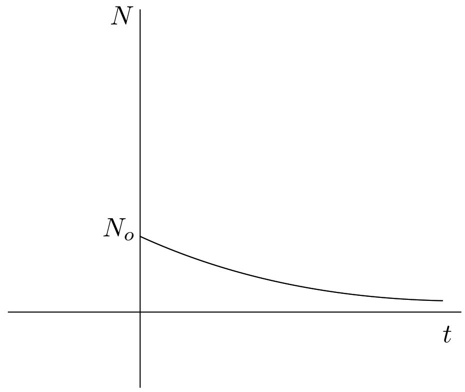

Figure 4.6.11: Sketch of graph of $N = N(t)$ for $a < 0$

(4.115) can also be used to obtain a expression for the *per–capita* growth rate in terms of the doubling time,

$$a = \frac{\ln 2}{\tau_2}. \tag{4.116}$$

Example 4.6.2. Certain strand of *E. Coli* bacteria has a division cycle that lasts about 20 minutes. We can use this time as a measure of the doubling time, τ_2. Then, using (4.116) and measuring the time in hours, we have that, under the assumption of Malthusian growth, the *per–capita* growth rate of a bacteria colony would be

$$a = \frac{\ln 2}{1/3} \doteq 2.08$$

in units of one per hour. Thus, the population size at time t (in hours) of the bacterial colony is

$$N(t) \doteq N_o e^{2.08t}, \tag{4.117}$$

where N_o is the size of the population at time $t = 0$.

Example 4.6.3. Suppose a population growing under a Malthusian growth model has a doubling time τ_2. We then have that

$$N(t) = N_o e^{at}, \tag{4.118}$$

where a is given by (4.115). Substituting the expression for a in (4.115) into (4.118) yields

$$\begin{aligned} N(t) &= N_o \exp\left(\frac{\ln 2}{\tau_2}\, t\right) \\ &= N_o \exp\left(\ln 2 \frac{t}{\tau_2}\right) \\ &= N_o[\exp(\ln 2)]^{t/\tau_2}, \end{aligned} \tag{4.119}$$

where we have used property (v) in Proposition 4.4.5. In then follows from the definition of the exponential in Definition 4.4.1 and the calculations in (4.119) that

$$N(t) = N_o 2^{t/\tau_2}, \quad \text{for } t \geqslant 0. \tag{4.120}$$

If we now measure time in units of doubling time (e.g., in units of division cycles for bacterial populations), we obtain from (4.120) that

$$N(\tau) = N_o 2^{\tau},$$

where $\tau = \dfrac{t}{\tau_2}$ measures the number of doubling times since time $t = 0$.

Example 4.6.4. Suppose that a single cell of the bacterium *E. coli* divides every twenty minutes. Given that the average mass of an *E. coli* bacterium is 10^{-12} grams, if a cell of *E. coli* was allowed to reproduce without restrain to produce a mega–colony, estimate the time that it would take for the total mass of the bacterial colony to be that of the earth (approximately 6×10^{24} Kg). (For this example, assume a Malthusian growth model.)

Solution: We use equation (4.117) derived in Example 4.6.2, with $N_o = 1$. We then have that the size of the colony at time t (in hours) is given by

$$N(t) \doteq e^{2.08t}, \tag{4.121}$$

for t hours after $t = 0$.

First, find the number, N, of bacteria that are needed to come up with the mass of the earth; that is we need to solve the equation

$$10^{-12} N = 6 \times 10^{27},$$

(in grams), which yields

$$N = 6 \times 10^{39}. \tag{4.122}$$

Next, we find the time t so that

$$N(t) = N,$$

or, using (4.121) and (4.122),

$$e^{2.08t} = 6 \times 10^{39},$$

which can be solved for t to yield

$$t \doteq \frac{\ln 6 + 39 \ln(10)}{2.08} \doteq 44 \text{ hours}.$$

Thus, it would take about a day and 20 hours (under 2 days) for the the total mass of the bacterial colony to be that of the earth. □

4.7 Testing the Malthusian Model

The calculations presented in Example 4.6.4 illustrate in a dramatic way the inadequacy of the Malthusian model to predict the growth of microorganisms. The results are not surprising given that the Malthusian model used in the calculations does not take into account competition for resources and availability of nutrients. The next step in the modeling is to account for the effects of nutrient concentration in a bacterial colony, for instance, in the growth rate of the bacteria.

Time (hours)	Concentration (OD_{650})
0.0	0.032
0.5	0.039
1.0	0.069
1.5	0.110
2.0	0.170
2.5	0.229
3.0	0.261
3.5	0.288
4.0	0.309
4.5	0.327
5.0	0.347

Table 4.3: *Staphylococcus aureus* Growth Data

Before we proceed with modifying the Malthusian model, in this section we illustrate how to test a model against actual experimental data. Table 4.3 displays data on the growth of *Staphylococcus aureus* collected by Segall, A. and Gunderson, C. in the Department of Biology, San Diego State University, 2004, (unpublished data[1]).

The second column in Table 4.3 shows optical density measurements (OD_{650}) that are proportional to the number of bacteria in the culture.

We have plotted the data in Table 4.3 in a scatter plot using MS Excel. The plot is shown in Figure 4.7.12. We see from the plot in Figure 4.7.12 that bacteria in the experiment do not seem to be undergoing exponential growth; in fact, the shape of a smooth curve going through the data points is similar to that of the s–shaped curve that characterizes logistic growth (see for instance the sketches in Figure 3.1.2). We will see in a later section how to fit the data in Table 4.3 to the logistic model in (2.16). In this section, though, we will illustrate the procedure that we will follow then to fit the data in Table 4.3 to the Malthusian model in (2.12). In addition to learning how to fit data to a mathematical model, we will also learn how far the data in Table 4.3 diverges from the predictions of the Malthusian model. We will also see that, at the beginning of the process (in

[1] see also `http://www-rohan.sdsu.edu/~jmahaffy/courses/f11/math122/index.html`

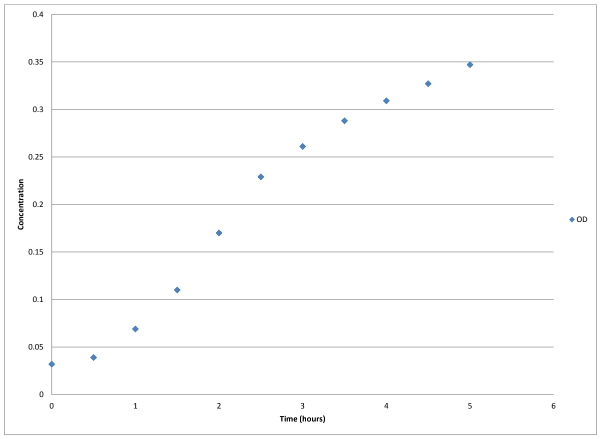

Figure 4.7.12: *Staphylococcus aureus* Growth Data Scatter Plot

particular, within the first two hours in the experiment), the Malthusian model provides a very good fit to the data.

When fitting the exponential model

$$N(t) = N_o e^{at}, \quad \text{for } t \in \mathbb{R}, \tag{4.123}$$

to a set of observations, it is convenient to consider the equation

$$\ln N = \ln(N_o) + at, \quad \text{for } t \in \mathbb{R}, \tag{4.124}$$

which is obtained by taking the natural logarithm on both sides of (4.123). It follows, then, that if the data are in accord with the Malthusian model, then a plot of $\ln N$ versus t should yield a scatter plot with a strong linear trend. For the case of the data in Table 4.3, we compute the natural logarithm of the optical density values in the second column to obtain the values shown in Table 4.4. Figure 4.7.13 shows a scatter plot of the data in Table 4.4. The plot in Figure 4.7.13 also shows the best fitting line through the data (least–squares fit), namely, the straight line with equation

$$y = -3.1263 + 0.4959\ t. \tag{4.125}$$

The value of $R^2 = 0.883$ is also displayed in Figure 4.7.13; R^2 gives a measure of how likely the linear model is able to predict future outcomes in the experiment. In this case, R^2 is 88.3%, which is not too bad of a percentage; however, it is not very good either. This is not surprising given the information provided in the plot in Figure 4.7.13; although there is an increasing trend in the data, the nature of the trend is not necessarily captured by the linear model. Nevertheless,

Time (hours)	$\ln(OD_{650})$
0.0	-3.442
0.5	-3.244
1.0	-2.674
1.5	-2.207
2.0	-1.772
2.5	-1.474
3.0	-1.343
3.5	-1.245
4.0	-1.174
4.5	-1.118
5.0	-1.058

Table 4.4: Log of Concentration for *Staphylococcus aureus* Growth Data

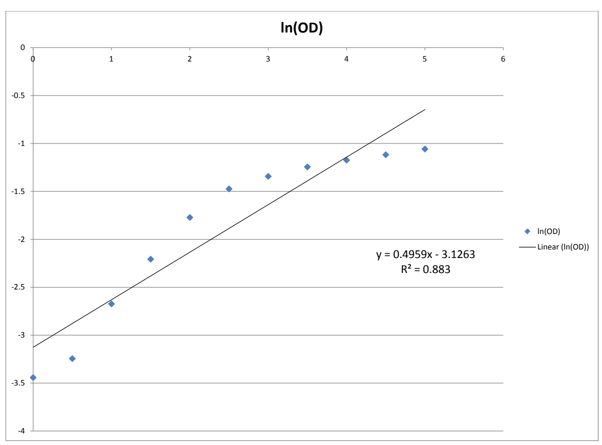

Figure 4.7.13: *Staphylococcus aureus* Log Growth Data Linear Fit

for the sake of illustration, we proceed with the process of fitting the data in (4.3) to the Malthusian model.

Comparing the least–squares fit line in (4.125) with the linear model in (4.124), we obtain estimates for $\ln C_o$, where C_o is the initial concentration of bacteria, and the *per–capita* growth rate, a, of the population as follows:

$$\ln C_o \doteq -3.1263 \tag{4.126}$$

and

$$a \doteq 0.4959. \tag{4.127}$$

From (4.126) we obtain that

$$C_o \doteq 0.044.$$

Thus, the Malthusian model yields the following predicted values for optical density measurements,

$$C(t) \doteq (0.044)e^{0.4959\ t}, \quad \text{for } t \geqslant 0. \tag{4.128}$$

The predicted values are computed from (4.128) for the times in the first column of Table 4.3 and plotted, using MS Excel, together with the observed values in the second column of the table. The resulting plot is shown in Figure 4.7.14. An examination of the plot in Figure 4.7.14 reveals that the Multhus fit of the data in Table 4.3 begins to diverge drastically from the observed values after the fourth hour in the experiment. The plot also shows that the Malthusian model provides a good fit to the data at the very beginning. In We will next illustrate this by fitting the first six data points in Table 4.3 to the Malthusian model; in other words, we will determine an exponential fit to the data in the following table A plot of the natural logarithm of the optical density measurements in the

Time (hours)	Concentration (OD_{650})
0.0	0.032
0.5	0.039
1.0	0.069
1.5	0.110
2.0	0.170
2.5	0.229

Table 4.5: *Staphylococcus aureus* Growth Data (first six observations)

second column of Table 4.5 versus time is shown in Figure 4.7.15. The plot in Figure 4.7.15 also shows with a least–squares linear fit of the natural logarithm of the second column of Table 4.5 versus t. We see that the R^2 value of this fit is 98.8%, which shows that the linear model in this case has a lot of predictive power. An examination of the plot also shows that the linear fit is very good. The least–squares linear fit shown in Figure 4.7.15 has the equation

$$y = -3.5205 + 0.8413\ t,$$

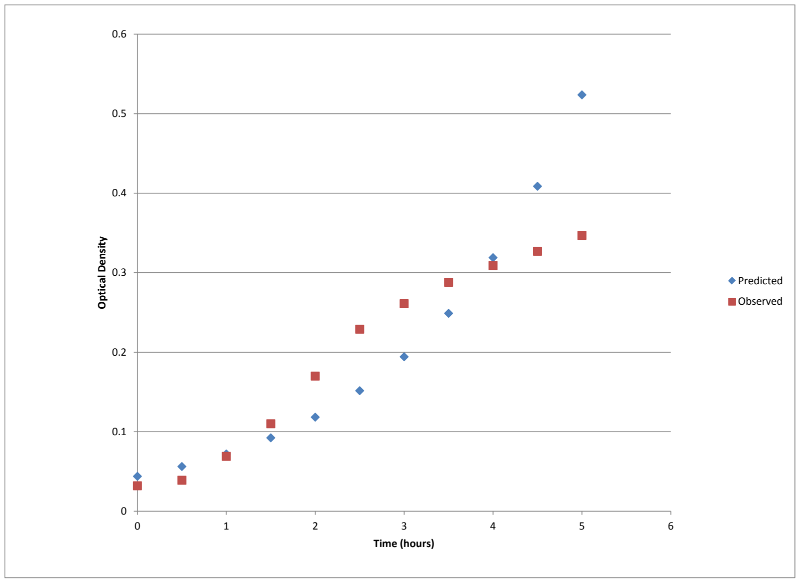

Figure 4.7.14: *Staphylococcus aureus* Fitting the Malthus Model to the Data in Table 4.3

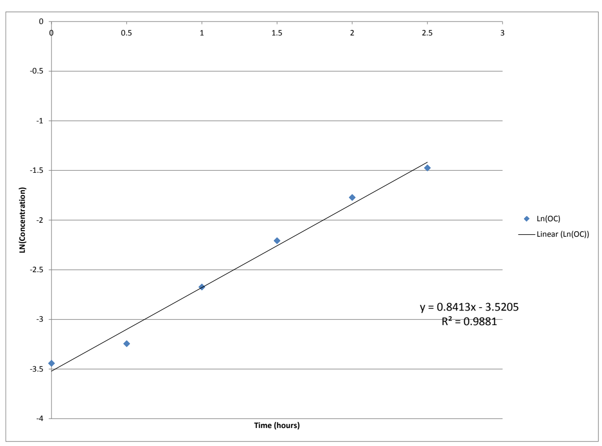

Figure 4.7.15: *Staphylococcus aureus* Linear Fit of the log of second column in Table 4.5 versus t

which gives us the estimates

$$\ln C_o \doteq -3.5205 \qquad (4.129)$$

and

$$a \doteq 0.8413. \qquad (4.130)$$

From (4.129) we obtain that

$$C_o \doteq 0.030. \qquad (4.131)$$

In this case we obtain, using (4.130) and (4.131), the following predicted values for optical density measurements,

$$C(t) \doteq (0.030)e^{0.8413\ t}, \quad \text{for } t \geqslant 0. \qquad (4.132)$$

A plot of the values predicted by (4.132) alongside the observed values in Table 4.5 is shown in We see in the plot in Figure 4.7.16 that the Malthusian model

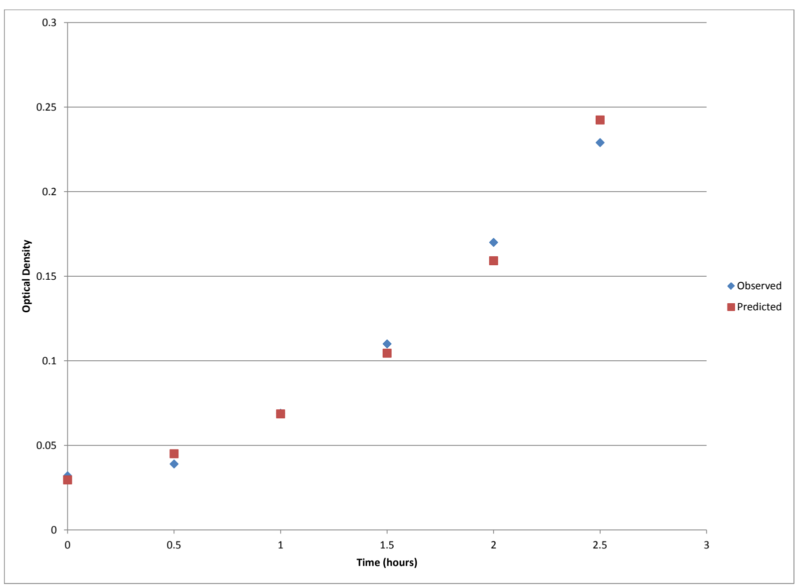

Figure 4.7.16: Malthus Fit of first six measurements in *Staphylococcus aureus* data

provides a very good fit to the first six data points in the *Staphylococcus aureus* data in Table 4.3.

4.8 Linear First Order Differential Equations

The differential equations

$$\frac{dQ}{dt} = c_i F_i - \frac{F_o}{V}\, Q.$$

and

$$\frac{dN}{dt} = aN,$$

derived in Sections 2.1 and 2.2, respectively, are exaples of first order linear differential equation. In general, a first order linear differential equation is of the form

$$\frac{dy}{dt} = a(t)y + b(t), \tag{4.133}$$

where $a = a(t)$ and $b = b(t)$ are continuous functions of t defined on some interval I. Note that the logistic equation

$$\frac{dN}{dt} = rN - \frac{r}{K}N^2,$$

also derived in Section 2.2, is not linear; it is, however, a first order differential equaition

In this section we will see how to obtain a general solution to the linear first order differential equation in (4.133). We will also show that if $a(t)$ and $b(t)$ are continuous on an open interval, I, $t_o \in I$, and $y_o \in \mathbb{R}$, then the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t) \\ y(t_o) = y_o, \end{cases} \tag{4.134}$$

has a unique solution $y = y(t)$ defined for $t \in I$.

4.8.1 Linear Equation with Constant Coefficients

We will first consider the case in which the coefficient functions a and b on the right–hand side of (4.133) are constant, and $a \neq 0$; in other words, we will first see how to solve the first order differential equation

$$\frac{dy}{dt} = ay + b, \tag{4.135}$$

where a and b are real numbers with $a \neq 0$.

Since $a \neq 0$, we can factor a on the right–hand side of (4.135) to obtain

$$\frac{dy}{dt} = a\left(y + \frac{a}{b}\right). \tag{4.136}$$

Next, set

$$\overline{y} = -\frac{b}{a}, \tag{4.137}$$

so that (4.136) becomes

$$\frac{dy}{dt} = a\,(y - \overline{y})\,. \tag{4.138}$$

We will show how to solve the differential equation in (4.138) by means of a method known as **separation of variables**. The idea behind the method of solutions is to re–write the differential equation as an integral equation where the variables y and t are on different sides of the equation. For the equation in (4.138), the corresponding integral equation is

$$\int \frac{1}{y - \overline{y}}\; dy = \int a\; dt. \tag{4.139}$$

Evaluating the indefinite integrals on each side of (4.139) yields

$$\ln|y - \overline{y}| = at + c_1, \tag{4.140}$$

for arbitrary constant c_1. We see that we can solve for y in (4.140) by, first, applying the exponential function on both sides of the equation in (4.140) to obtain

$$|y - \overline{y}| = c_2 e^{at}, \tag{4.141}$$

where we have set $c_2 = e^{c_1}$; then, re–write the equation in (4.141) as

$$|e^{-at}(y - \overline{y})| = c_2, \quad \text{ for all } t \in \mathbb{R}. \tag{4.142}$$

Finally, as a consequence of the continuity of the exponential function and of y, we obtain from (4.142) that

$$e^{-at}(y - \overline{y}) = c, \quad \text{ for all } t \in \mathbb{R}, \tag{4.143}$$

and an arbitrary constant c; we can then solve (4.143) for $y = y(t)$ to obtain

$$y(t) = \overline{y} + ce^{at}, \quad \text{ for all } t \in \mathbb{R}. \tag{4.144}$$

The expression in (4.144) gives what is known as the **general solution** to the differential equation in (4.135); it gives all possible solutions of the equation obtained by the method of separation of variables in terms of the parameter c. In order to find a solution of (4.135) which satisfies the initial condition

$$y(t_o) = y_o, \tag{4.145}$$

we find the value of c in (4.144) by solving the equation

$$\overline{y} + ce^{at_o} = y_o$$

to obtain

$$c = (y_o - \overline{y})e^{-at_o}. \tag{4.146}$$

Substituting the value for c in (4.146) into (4.144) yields the function

$$y(t) = \overline{y} + (y_o - \overline{y})e^{a(t-t_o)}, \quad \text{ for all } t \in \mathbb{R}. \tag{4.147}$$

We will presently show the the function $y = y(t)$ given in (4.147), where $\overline{y}$ is defined in (4.137), is the only solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ay + b; \\ y(t_o) = y_o, \end{cases} \tag{4.148}$$

for $a, b \in \mathbb{R}$, with $a \neq 0$. In fact, let $v = v(t)$ denote any solution to the initial value problem (4.148); then,

$$v'(t) = av(t) + b, \quad \text{for all } t \in \mathbb{R}, \tag{4.149}$$

and

$$v(t_o) = y_o. \tag{4.150}$$

Define $w\colon \mathbb{R} \to \mathbb{R}$ by

$$w(t) = [v(t) - \overline{y}]e^{-a(t-t_o)}, \quad \text{for all } t \in \mathbb{R}. \tag{4.151}$$

The function, w, defined in (4.151) is differentiable, and its derivative, by virtue of the product rule and the Chain Rule, is

$$\begin{aligned} v'(t) &= v'(t)e^{-a(t-t_o)} - a[v(t) - \overline{y}]e^{-a(t-t_o)} \\ &= [v'(t) - av(t) + a\overline{y}]e^{-a(t-t_o)} \\ &= [b - b]e^{-a(t-t_o)} \\ &= 0, \end{aligned} \tag{4.152}$$

for all $t \in \mathbb{R}$, where we have used (4.149) and (4.137). It follows from the result of the calculations in (4.152) and Problem 1 in Assignment #1 that

$$w(t) = c, \quad \text{for all } t \in \mathbb{R}, \tag{4.153}$$

and some constant c. We can find the value of c in (4.153) by using the initial condition in (4.150) to obtain, using (4.151) that

$$c = w(t_o) = [v(t_o) - \overline{y}]e^{-a(t_o-t_o)} = y_o - \overline{y}. \tag{4.154}$$

Combining (4.154), (4.153) and (4.151) yields

$$[v(t) - \overline{y}]e^{-a(t-t_o)} = y_o - \overline{y}, \quad \text{for all } t \in \mathbb{R},$$

from which we get that

$$v(t) = \overline{y} + (y_o - \overline{y})e^{a(t-t_o)}, \quad \text{for all } t \in \mathbb{R},$$

which shows that the function given in (4.147) is the only solution to initial value problem in (4.148). Hence, the initial value problem in (4.148) has a unique solutions which is defined for all values of $t \in \mathbb{R}$.

Example 4.8.1 (Newton's Law of Cooling)**.** Let $u = u(t)$ denote the temperature of an object in an environment with fixed temperature $\overline{u}$. If the object's temperature is higher than the ambient temperature, $\overline{u}$, heat energy will be transferred from the object to the environment causing the temperature of the object, $u(t)$, to decrease with time. In some situations, the cooling of the object can by described by Newton's Law of Cooling, which states that the rate of change of the object's temperature is proportional to the difference between the objects's temperature and that of the surrounding environment. Assuming that the ambient's temperature remains constant, Newton's Law of Cooling can be written in symbols as

$$\frac{du}{dt} = -k(u - \overline{u}), \tag{4.155}$$

where k denotes a (positive) constant of proportionality. Note that the equation in (4.155) also describes the rate of heating of the object in case that the object's temperature is smaller than the ambient temperature.

The equation in (4.155) is a linear differential equation with constant coefficients of the type we have been studying in this section. We can solve it using separation of variables to obtain

$$u(t) = \overline{u} + ce^{-kt}, \quad \text{for all } t, \tag{4.156}$$

where c is a constant. If the temperature of the object at time $t = 0$ is known, say $u(0) = u_o$, we obtain from (4.156) that

$$u(t) = \overline{u} + (u_o - \overline{u})e^{-kt}, \quad \text{for all } t \in \mathbb{R}. \tag{4.157}$$

Figure 4.8.17 shows sketches of graphs of possible solutions of (4.155) for various initial temperatures.

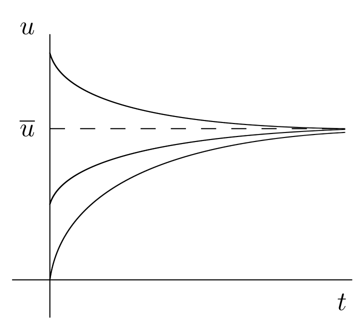

Figure 4.8.17: Sketch of graph of $u = u(t)$

4.8.2 Qualitative Analysis of the Linear First Order Equation

The value $\overline{y}$ given in (4.137) is called an equilibrium solution to the equation

$$\frac{dy}{dt} = ay + b,$$

for $a \neq 0$, or equivalently, of the equation

$$\frac{dy}{dt} = a(y - \overline{y}). \tag{4.158}$$

In general, given a noncontinuous function, $g\colon \mathbb{R} \to \mathbb{R}$, an equilibrium point of the differential equation

$$\frac{dy}{dt} = g(y) \tag{4.159}$$

is a solution of the equation

$$g(y) = 0.$$

We saw in Section 4.8 that the solution to the differential equation in (4.158) satisfying the initial condition $y(0) = y_o$ is given by (4.147); namely,

$$y(t) = \overline{y} + (y_o - \overline{y})e^{at}, \quad \text{for all } t \in \mathbb{R}. \tag{4.160}$$

From (4.160) we obtain

$$|y(t) - \overline{y}| = |y_o - \overline{y}|e^{at}, \quad \text{for all } t \in \mathbb{R}. \tag{4.161}$$

Since we are assuming that $a \neq 0$, there are two possible behaviors that the solution $y = y(t)$ can have around the equilibrium point $\overline{y}$. First, if $a > 0$ we see from (4.161) that, for $y_o \neq \overline{y}$, the distance from $y(t)$ to $\overline{y}$ increases as t increases. In this case we say that the equilibrium point, $\overline{y}$, is **unstable**. In general, we say that an equilibrium point, $\overline{y}$, is unstable when solutions to the differential equation in (4.159) that begin near $\overline{y}$ tend away from $\overline{y}$.

On the other hand, if $a < 0$, we see from (4.161) that $|y(t) - \overline{y}|$ decreases as t increases; so that, if $a < 0$, solutions to the differential equation that begin near $\overline{y}$ will tend towards $\overline{y}$. It this case we say that $\overline{y}$ is **stable**. In fact, not only is $y(t)$ tending closer to $\overline{y}$ as t increases, but, in the limit at $t \to \infty$, $y(t)$ will tend to $\overline{y}$ as seen from (4.161); namely,

$$\lim_{t \to \infty} |y(t) - \overline{y}| = 0, \tag{4.162}$$

since $a < 0$. When (4.162) happens we say that $\overline{y}$ is **asymptotically stable**.

4.8.3 Linear Equations with Variable Coefficients

We continue with the analysis of the the general linear first order equation in (4.133); namely,

$$\frac{dy}{dt} = a(t)y + b(t), \tag{4.163}$$

where $a = a(t)$ and $b = b(t)$ are continuous functions of t defined on some interval I. We first consider the following example.

Example 4.8.2. Find the general solution to the first order differential equation

$$\frac{dy}{dt} = y + t, \tag{4.164}$$

and then find a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= y + t; \\ y(0) &= 0. \end{cases} \tag{4.165}$$

The method of separation of variables does not apply to the equation in (4.164); however, we can rewrite it in the form

$$\frac{dy}{dt} - y = t. \tag{4.166}$$

Next, multiply both sides of the equation in (4.166) by e^{-t} to obtain

$$e^{-t}\frac{dy}{dt} - e^{-t}y = te^{-t}. \tag{4.167}$$

Note that, by virtue of the product rule, the left–hand side of the equation in (4.167) can be written as

$$\frac{d}{dt}\left[e^{-t}y\right],$$

so that the equation in (4.167) can now be written in the form

$$\frac{d}{dt}\left[e^{-t}y\right] = te^{-t}. \tag{4.168}$$

Thus, even though we were not able to separate variables in the equation in (4.164), by introducing the factor e^{-t}, the equation can be written in a form that can be solved for y by integration. In fact, integrating on on both sides of (4.168) we obtain that

$$e^{-t}y = \int te^{-t}\ dt + c, \tag{4.169}$$

where c is an arbitrary constant. Denoting the indefinite integral on the right–hand side of (4.169) by $W(t)$, we obtain

$$e^{-t}y = W(t) + c, \tag{4.170}$$

We may take $W(t)$ to be

$$W(t) = \int_0^t \tau e^{-\tau}\ d\tau, \quad \text{for all } t, \tag{4.171}$$

so that

$$W(0) = 0. \tag{4.172}$$

Solving (4.170) for y we obtain the general solution

$$y(t) = c\, e^t + e^t W(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.173}$$

Using the initial condition $y(0) = 0$ we obtain from (4.173) that

$$c + W(0) = 0,$$

so that, using (4.172), $c = 0$ and therefore, by virtue of (4.173),

$$y(t) = e^t W(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.174}$$

solves the initial value problem in (4.165).

4.8.4 Integration by Parts

To complete the solution to the initial value problem (4.165) in Example 4.8.2, we need to evaluate the integral defining $W(t)$ in (4.172). We do this by employing an integration technique known as **integration by parts**. This technique is a consequence of the product rule: Suppose $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are two functions which are differentiable on an open interval I, then the product function $fg\colon I \to \mathbb{R}$ is differentiable on I and

$$\frac{d}{dt}[f(t)g(t)] = f(t)g'(t) + f'(t)g(t), \quad \text{for all } t \in I. \tag{4.175}$$

Next, integrate with respect to t on both sides of (4.175) and use the Fundamental Theorem of Calculus to obtain

$$f(t)g(t) = \int f(t)g'(t)\ dt + f'(t)g(t)\ dt,$$

from which we obtain

$$\int f(t)g'(t)\ dt = f(t)g(t) - \int g(t)f'(t)\ dt. \tag{4.176}$$

The formula in (4.176) allows one to express the integral on the left–hand side, which might be difficult to evaluate by hand, in terms of another integral, which might be easier to do. We will illustrate the use of the formula in (4.176) when evaluating the integral defining $W(t)$ in (4.170). Before we do so, though, we will rewrite the formula in terms of differentials $du = f'(t)\ dt$ and $dv = g'(t)\ dt$, where we have set $u = f(t)$ and $v = g(t)$. In terms of the new variables and differentials, the formula in (4.176) becomes

$$\int u\ dv = uv - \int v\ du. \tag{4.177}$$

The formula in (4.177) is easier to remember, and is also easier to apply.

Example 4.8.3. Evaluate the indefinite integral $\int te^{-t}\, dt$.

Solution: Set

$$u = t \quad \text{and} \quad dv = e^{-t}\, dt;$$

$$\text{then} \quad du = dt \quad \text{and} \quad v = -e^{-t},$$

so that, applying the integration by parts formula in (4.177),

$$\int te^{-t}\, dt = -te^{-t} + \int e^{-t}\, dt. \tag{4.178}$$

Next, evaluate the integral on the right–hand side of (4.178) to get

$$\int te^{-t}\, dt = -te^{-t} - e^{-t} + c, \tag{4.179}$$

where the constant c is arbitrary. □

Using the result in Example 4.8.3 we can evaluate the function $W(t)$ defined in (4.172) to obtain

$$\begin{aligned} W(t) &= \int_0^t \tau e^{-\tau}\, d\tau \\ &= \left[-\tau e^{-\tau} - e^{-\tau}\right]_0^t \\ &= 1 - te^{-t} - e^{-t}, \end{aligned}$$

so that, in view of (4.174),

$$y(t) = e^t W(t) = e^t - t - 1, \quad \text{for all } t \in \mathbb{R},$$

solves the initial value problem in (4.165). In the following section we will show that this is the only solution to the initial value problem.

4.8.5 Integrating Factors

In Example 4.8.2 we saw how to go from the first–order differential equation

$$\frac{dy}{dt} = y + t \tag{4.180}$$

to the equation

$$\frac{d}{dt}\left[e^{-t}y\right] = te^{-t} \tag{4.181}$$

by multiplying the equation

$$\frac{dy}{dt} - y = t \tag{4.182}$$

by e^{-t}. The function $\mu(t) = e^{-t}$ is an example of an **integrating factor**. Multiplying the equation in (4.182) by $\mu(t)$ allows one to write the equation in

(4.180) in the form in (4.181), which can be integrated and then solved for y. In this section we see that the method of finding an integrating factor can be used to find a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t); \\ y(t_o) = y_o, \end{cases} \tag{4.183}$$

where $a = a(t)$ and $b = b(t)$ are continuous functions defined in an open interval, I, with $t_o \in I$ and $y_o \in \mathbb{R}$.

Rewrite the differential equation in (4.183) as

$$\frac{dy}{dt} - a(t)y = b(t), \tag{4.184}$$

and multiply both sides of the equation in (4.184) by

$$\mu(t) = \exp(-A(t)), \quad \text{for } t \in I, \tag{4.185}$$

where A is an antiderivative of a; that is,

$$A'(t) = a(t), \quad \text{for all } t \in I, \tag{4.186}$$

to obtain from (4.184) that

$$\mu(t)\frac{dy}{dt} - a(t)\mu(t)y = \mu(t)b(t), \quad \text{for all } t \in I. \tag{4.187}$$

Now, it follows from (4.185) and (4.186) that

$$\mu'(t) = -a(t)\mu(t), \quad \text{for all } t \in I. \tag{4.188}$$

Thus, the equation in (4.187) may be written as

$$\mu(t)\frac{dy}{dt} + \mu'(t)y = \mu(t)b(t), \quad \text{for all } t \in I,$$

which, by virtue of the product rule, may in turn be written as

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)b(t), \quad \text{for all } t \in I. \tag{4.189}$$

The equation in (4.189) may now be integrated with respect to t and then solved for y to obtain the general solution to the differential equation in (4.183).

Example 4.8.4. Find the general solution to the differential equation

$$\frac{dy}{dt} = -\frac{2}{t}\, y + e^{-t}, \quad \text{for } t > 0. \tag{4.190}$$

Solution: Rewrite the equation in (4.190) as

$$\frac{dy}{dt} + \frac{2}{t}\, y = e^{-t}, \qquad \text{for } t > 0. \tag{4.191}$$

In this case, an integrating factor is provided by

$$\begin{aligned} \mu(t) &= \exp\left(\int \frac{2}{t}\, dt\right) \\ &= \exp\left(2\ln t\right) \\ &= \exp\left(\ln(t^2)\right) \\ &= t^2, \end{aligned}$$

for $t > 0$. Thus, multiplying on both sides of (4.191) by $\mu(t) = t^2$, for $t > 0$, yields

$$t^2\frac{dy}{dt} + 2ty = t^2e^{-t}, \qquad \text{for } t > 0.$$

which can be written as

$$\frac{dy}{dt}\left[t^2y\right] = t^2e^{-t}, \qquad \text{for } t > 0, \tag{4.192}$$

by virtue of the product rule.

Next, integrate (4.192) with respect to t to get

$$t^2y = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + c, \qquad \text{for } t > 0, \tag{4.193}$$

and for arbitrary c, where we have used integration by parts (twice) to obtain the antiderivative on the right–hand side of (4.193).

Finally, solving for y in (4.193) yields the general solution

$$y(t) = -e^{-t} - \frac{2}{t}\, e^{-t} - \frac{2}{t^2}e^{-t} + \frac{c}{t^2}, \qquad \text{for } t > 0. \tag{4.194}$$

□

In order to obtain a solution to the initial value problem in (4.183), it is convenient to choose A so that

$$A(t_o) = 0; \tag{4.195}$$

in other words

$$A(t) = \int_{t_o}^{t} a(\tau)\, d\tau, \qquad \text{for all } t \in I.$$

It then follows from (4.185) and (4.195) that

$$\mu(t_o) = 1. \tag{4.196}$$

Next, integrate the differential equation in (4.189) from t_o to t in I to obtain

$$\int_{t_o}^{t} \frac{d}{d\tau}\left[\mu(\tau)y(\tau)\right]\ d\tau = \int_{t_o}^{t} \mu(\tau)b(\tau)\ d\tau. \tag{4.197}$$

The integral on the left–hand side of (4.197) can be evaluated using the Fundamental Theorem of Calculus to yield

$$\mu(t)y(t) - \mu(t_o)y(t_o) = \int_{t_o}^{t} \mu(\tau)b(\tau)\ d\tau, \quad \text{for } t \in I. \tag{4.198}$$

Using the initial condition in (4.183) and (4.196), we obtain from (4.198) that

$$\mu(t)y(t) = y_o + \int_{t_o}^{t} \mu(\tau)b(\tau)\ d\tau, \quad \text{for } t \in I. \tag{4.199}$$

We can now solve of $y(t)$ in (4.199) by dividing by $\mu(t) = \exp(-A(t))$ to obtain

$$y(t) = y_o \exp(A(t)) + \exp(A(t)) \int_{t_o}^{t} \exp(-A(\tau))b(\tau)\ d\tau, \quad \text{for } t \in I. \tag{4.200}$$

It can be verified, through application of the Fundamental Theorem of Calculus, the Chain Rule and the product rule, that the function $y = y(t)$ defined in (4.200) solves the initial value problem in (4.183) for the linear first order equation in (4.163) for continuous coefficients $a = a(t)$ and $b = b(t)$ defined for $t \in I$. It can also be proved by techniques similar to the ones discussed in these notes that the formula in (4.200) provides the only solution to the initial value problem in (4.183).

4.9 Solving the Logistic Equation

In this section we compute a solution to the initial value problem for the Logistic equation

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right); \\ \\ N(0) = N_o, \end{cases} \tag{4.201}$$

for $N_o > 0$. We will present two ways to solve the equation in (4.201). The first way involves a change of variables that allows us to relate a solution of the initial value problem in (4.201) that of of an initial value problem for a linear first order equation. Thus, in the firs method of solution we will be able to use the results that we developed in the previous section for linear differential equations. In particular, we will be able to use the uniqueness of the solution for the initial value problem for the linear equation to conclude that the initial value problem in (4.201) has a unique solution for each $N_o > 0$.

The second method for solving the logistic equation involves separation of variables and an integration technique known as **partial fractions**. This solution technique does not yield uniqueness; however, it can be applied to large class of differential equations.

4.9.1 Existence and Uniqueness

We look for a solution of the initial value problem in (4.201) for the case $N_o > 0$. Suppose for the moment that we have found that solution and denote it by $N = N(t)$; then,

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) \tag{4.202}$$

for t in some interval of existence, J, which contains 0; and

$$N(0) = N_o \tag{4.203}$$

Since we are assuming that $N_o > 0$, we may also assume that $N(t) > 0$ for t in some portion of J that contains 0; denote that portion by I, so that

$$N(t) > 0, \qquad \text{for } t \in I. \tag{4.204}$$

Define a new function $u = u(t)$, for $t \in I$, by

$$u(t) = \frac{1}{N(t)}, \qquad \text{for } t \in I. \tag{4.205}$$

Then, $u = u(t)$ is differentiable in I and, by the Chain Rule,

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{[N(t)]^2}\, N'(t) \\ &= -\frac{1}{[N(t)]^2}\, rN(t)\left(1 - \frac{N(t)}{K}\right) \\ &= -r\left(\frac{1}{N(t)} - \frac{1}{K}\right), \end{aligned} \tag{4.206}$$

where we have used (4.202). It follows from the calculation in (4.206) and the definition of $u = u(t)$ in (4.205) that u solves the differential equation

$$\frac{du}{dt} = -r\left(u - \frac{1}{K}\right). \tag{4.207}$$

The equation in (4.207) is a linear first–order equation with constant coefficients that has general solution

$$u(t) = \frac{1}{K} + c\, e^{-rt}, \tag{4.208}$$

for arbitrary c, which is defined for all values of t. Using the initial condition in (4.203) we see from (4.208) that

$$\frac{1}{K} + c = \frac{1}{N_o},$$

which yields that

$$c = \frac{1}{N_o} - \frac{1}{K}. \tag{4.209}$$

Substituting the value for c in (4.209) into (4.208) yields

$$u(t) = \frac{1}{K} + \left(\frac{1}{N_o} - \frac{1}{K}\right) e^{-rt}. \tag{4.210}$$

The function $u = u(t)$ defined in (4.210) is the unique solution to the initial value problem

$$\begin{cases} \dfrac{du}{dt} = -r\left(u - \dfrac{1}{K}\right); \\ \\ N(0) = \dfrac{1}{N_o}, \end{cases} \tag{4.211}$$

for $N_o > 0$.

From (4.210) and (4.205) we obtain the following formula for $N(t)$:

$$N(t) = \frac{1}{u(t)} = \frac{N_o K}{N_o + (K - N_o)e^{-rt}}. \tag{4.212}$$

We can verify directly that the function $N = N(t)$ given by the formula in (4.212) does satisfy the differential equation in the initial value problem (4.201). It is also not hard to see that $N(0) = N_o$. Thus, the function $N = N(t)$ solves the initial value problem (4.201).

We observe from (4.212) that for the case in which $0 < N_o < K$, the function $N = N(t)$ given in (4.212) is defined for all values of $t \in \mathbb{R}$. To see why this assertion is true, observe that, since $N_o < K$, $K - N_o > 0$ so that

$$N_o + (K - N_o)e^{-rt} > 0, \text{ for all } t \in \mathbb{R};$$

that is, the denominator of the expression in (4.212) defining $N(t)$ is never 0.

On the other hand, for the case in which $N_o > K$, the denominator of the expression in (4.212) is zero when

$$N_o = (N_o - K)e^{-rt}, \tag{4.213}$$

or, solving for t in (4.213),

$$t = \frac{1}{r} \ln\left(\frac{N_o - K}{N_o}\right). \tag{4.214}$$

Note that the time t given in (4.214) is negative since we are assuming that $N_o > K$, so that $N_o - K < N_o$ (recall that the carrying capacity, K, is positive) and therefore $\dfrac{N_o - K}{N_o} < 1$. Thus, for the case in which $N_o > K$, the solution to the initial value problem (4.201) does not exists for the negative value of t given in (4.214); however, it does exits for

$$t > \frac{1}{r} \ln\left(\frac{N_o - K}{N_o}\right). \tag{4.215}$$

To see why this assertion is true, observe that if (4.215), then

$$e^{rt} > \frac{N_o - K}{N_o},$$

from which we get that

$$N_o > (N_o - K)e^{-rt},$$

which implies that

$$N_o + (K - N_o)e^{-rt} > 0;$$

that is, the denominator of the expression in (4.212) defining $N(t)$ is positive for $t > \ln[(N_o - K)/N_o]/r$. Hence, $N(t)$ is defined for all $t > \ln[(N_o - K)/N_o]/r$. Thus, we have shown that, for $N_o > 0$, the function defined by

$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}} \tag{4.216}$$

is defined in some interval containing 0 and all $t > 0$. This function solves the initial value problem in (4.201) and satisfies

$$\lim_{t\to\infty} N(t) = K,$$

since $r > 0$.

We next see that the function defined in (4.216) is the unique solution to the initial value problem (4.201). Suppose to the contrary that there are two distinct solutions, N_1 and N_2, of the initial value problem in (4.201). Since we are assuming that $N_o > 0$, we may also assume that $N_1(t) > 0$ and $N_2(t) > 0$ for t in some open interval, I, which contains 0. Then the functions u_1 and u_2 defined by

$$u_1(t) = \frac{1}{N_1(t)} \quad \text{and} \quad u_2(t) = \frac{1}{N_2(t)}, \quad \text{for } t \in I,$$

give rise to two distinct solutions to the initial value problem in (4.211). This is impossible since we showed in Section 4.8.1 that that problem has a unique solution. Hence, N_1 and N_2 cannot be distinct.

4.9.2 Partial Fractions

In this section we present another way to solve the Logistic equation. We start out by separating variables

$$\int \frac{1}{N(N-K)}\, dN = -\int \frac{r}{K}\, dt. \tag{4.217}$$

We evaluate the integral on the left–hand side of (4.217) by using an integration technique known as partial fractions. The idea for this technique is to write the integrand as a sum of fractions

$$\frac{1}{N(N-K)} = \frac{A}{N} + \frac{B}{N-K}, \tag{4.218}$$

where the constants A and B need to be determined so that the equation in (4.218) holds true. Assuming that we have found constants A and B so that (4.218) is true, we can then evaluate the integral on the left–hand side of (4.217) as follows:

$$\int \frac{1}{N(N-K)}\, dN = A\ln|N| + B\ln|N-K| + c, \tag{4.219}$$

for arbitrary constant c.

In order to find the constants A and B, first multiply both sides of the equation in (4.218) by $N(N-K)$ to get

$$1 = A(N-K) + BN,$$

or

$$0N + 1 = (A+B)N - AK. \tag{4.220}$$

Note that the right–hand side of the equation in (4.220) is polynomial in N. The constant, 1, in the left–hand side of (4.220) can also be thought of as a polynomial in N when written as $0N+1$. Two polynomials are equal if and only if corresponding coefficients are equal. Hence, the equality in (4.220) implies that

$$\left\{ \begin{array}{rcl} A+B & = & 0 \\ -AK & = & 1. \end{array} \right. \tag{4.221}$$

solving the second equation in (4.221) for A yields

$$A = -\frac{1}{K}. \tag{4.222}$$

Substituting the value for A in (4.222) into the first equation in (4.221) and solving for B yields

$$B = \frac{1}{K}. \tag{4.223}$$

Having determined the values for A and B in (4.222) and (4.223), respectively, we can evaluate the integral on the left–hand side of (4.217) by means of (4.219). We therefore obtain from (4.219) that

$$-\frac{1}{K}\ln|N| + \frac{1}{K}\ln|N-K| = -\frac{rt}{K} + c_1, \tag{4.224}$$

for some arbitrary constant c_1. Multiplying the equation in (4.224) and simplifying yields

$$\ln\left(\frac{|N-K|}{|N|}\right) = -rt + c_2, \tag{4.225}$$

for some arbitrary constant c_2. Applying the exponential function on both sides of (4.225) yields and simplifying yields

$$\frac{|N-K|}{|N|} = c_3 e^{-rt}, \tag{4.226}$$

where we have set $c_3 = e^{c_2}$. Next, use the continuity of the exponential function and of N, to conclude that

$$\frac{N-K}{N} = c\; e^{-rt}, \tag{4.227}$$

for some constant c. Solving for N in (4.227) we obtain the general solution to the Logistic differential equation:

$$N(t) = \frac{K}{1 - c\; e^{-rt}}. \tag{4.228}$$

In order to compute a solution to the logistic equation satisfying the initial condition $N(0) = N_o$, for $N_o \neq 0$, we first determine the value of c from (4.226) to obtain

$$c = \frac{N_o - K}{N_o}. \tag{4.229}$$

Substituting the value of c in (4.229) into (4.228) and simplifying yields

$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}},$$

which is the same formula given in (4.216) for the solution to the initial value for the Logistic equation in (4.201).

The procedure described so far yields existence; however, we can not conclude uniqueness from it. Nevertheless, the procedure applies to equations other than the Logistic equation.

Example 4.9.1. Solve the initial value problem

$$\frac{dy}{dt} = 1 - y^2, \qquad y(0) = 2. \tag{4.230}$$

Solution: First, rewrite the differential equation in (4.230) as

$$\frac{dy}{dt} = -(y^2 - 1),$$

and the separate variables to get

$$\int \frac{1}{y^2 - 1}\; dy = -\int dt. \tag{4.231}$$

In order to evaluate the integral on the left–hand side of (4.231), first we factor the denominator in the integrand to get

$$\frac{1}{y^2 - 1} = \frac{1}{(y+1)(y-1)}. \tag{4.232}$$

We decompose the right–hand side in (4.232) by means of partial fractions as

$$\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}, \tag{4.233}$$

where the constants A and B are to be determined. Once A and B are determined, the integral on the left–hand side of (4.231) can be evaluated by virtue of (4.232) and (4.233) to obtain

$$\int \frac{1}{y^2 - 1}\, dy = A \ln|y+1| + B \ln|y-1| + c, \tag{4.234}$$

for arbitrary constant c.

In order to determine A and B, multiply on both sides of the equation in (4.233) by $(y+1)(y-1)$ to obtain

$$1 = A(y-1) + B(y+1),$$

or

$$0y + 1 = (A+B)y + B - A. \tag{4.235}$$

Equating corresponding coefficients for the polynomials on the each side of (4.235) yields the system

$$\begin{cases} A + B &= 0 \\ B - A &= 1. \end{cases} \tag{4.236}$$

Solving the system in (4.236) yields

$$A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}. \tag{4.237}$$

Substituting the values for A and B in (4.237) into (4.234) yields the left–hand side of (4.231) so that, integrating both sides of (4.231),

$$-\frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| = -t + c_1, \tag{4.238}$$

for arbitrary constant c_1. Next, multiply on both sides of (4.238) and simplify to get

$$\ln\left(\frac{|y-1|}{|y+1|}\right) = -2t + c_2, \tag{4.239}$$

for arbitrary constant c_2. Apply the exponential function on both sides of (4.239) to obtain

$$\frac{|y-1|}{|y+1|} = c_3\, e^{-2t}, \tag{4.240}$$

where we have set $c_3 = e^{c_2}$. Using the continuity of y and the exponential function we get from (4.240) that

$$\frac{y-1}{y+1} = c\, e^{-2t}, \tag{4.241}$$

for arbitrary constant c. Solving for y in (4.240) yields the general solution,

$$y(t) = \frac{1 + c\, e^{-2t}}{1 - c\, e^{-2t}}, \tag{4.242}$$

for the differential equation in (4.230). Using the initial condition in (4.230) in (4.241), we get

$$c = \frac{1}{3}. \tag{4.243}$$

Substituting the value of c in (4.243) into (4.242) yields a solution to the initial value problem in (4.230) given by

$$y(t) = \frac{3 + e^{-2t}}{3 - e^{-2t}}.$$

□

4.10 Testing the Logistic Model

In this section we fit bacterial growth data that we discussed presented in Section 4.7 to the Logistic model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right). \tag{4.244}$$

In this case we have two parameters, r and K, that we need to estimate. We have shown that K is the limiting value of $N(t)$ as $t \to \infty$; so, in practice, it might not be feasible to observe a population for a very long time until some kind of approximation to the limiting value can be obtained. Since, all estimates are based on finite time measurements, there is still no guarantee that the estimate will be close to the actual limiting value. Thus, we need to find another way to estimate the parameters based on a finite set of data like the one for bacterial growth given in Table 4.3 on page 46 in these notes.

The idea is to rewrite the model in (4.244) it terms of the *per–capita* growth rate

$$\frac{1}{N}\frac{dN}{dt} = r - \frac{r}{K}\ N. \tag{4.245}$$

The equation in (4.245) states that, in the Logistics model, there is a linear relation between the *per–capita* growth rate and the population size. Thus, in a scatter plot of *per–capita* growth rate versus N the data points should line up along line with slope $-\dfrac{r}{K}$ and y–intercept r. Hence, a least–squares fit of the data should provide estimates for r and K.

Since data in Table 4.3 on page 46 corresponds to a discrete set of points, we need to make the differential equation in (4.245) into a discrete equation

$$\frac{1}{N}\frac{\Delta N}{\Delta t} = r - \frac{r}{K}\ N, \tag{4.246}$$

where

$$\Delta N = N(t_i) - N(t_{i-1}), \quad \text{for } i = 1, 2, \ldots, m,$$

for the times $t_o, t_1, t_2, \ldots, t_m$ listed on the first column of Table 4.3; $N(t_i)$, for $i = 0, 1, 2, \ldots$, are the corresponding optical density values; and

$$\Delta t = t_i - t_{i-1} = 0.5, \quad \text{for } i = 1, 2, \ldots, m.$$

We can then rewrite the equation in (4.246) as

$$\frac{1}{N(t_{i-1})}\frac{N(t_i) - N(t_{i-1})}{0.5} = r - \frac{r}{K}\,N(t_i), \quad \text{for } i = 1, 2, \ldots, m. \tag{4.247}$$

The relative growth rate values on the left–hand side of (4.247) can be computed using a spreadsheet program (for instance, MS Excel), and plotted against the discrete set of density values, $N(t_i)$. Table 4.6 show values of *per–capita* growth rate and population density based on the *Staphylococcus aureus* growth data in Table 4.3. We discarded the data point corresponding to optical density of 0.039

Concentration (OD_{650})	*per–capita* Growth Rate
0.069	1.538
0.110	1.188
0.170	1.091
0.229	0.694
0.261	0.279
0.288	0.207
0.309	0.146
0.327	0.117
0.347	0.122

Table 4.6: *Staphylococcus aureus* Relative Growth Data

since it did fit into the linear pattern of the rest of the points shown in Figure 4.10.18. Keep in mind that our goal is to estimate the parameters r and K in equation (4.244) in order to obtain a formula for $N(t)$ given by the solution to the equation (4.244) subject to the initial condition $N(0) = N_o$; namely,

$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}}, \tag{4.248}$$

Figure 4.10.18 also show the least–squares linear fit for the data in Table 4.6) the corresponding equation of the line is

$$y = 1.878 - 5.459x.$$

Thus,

$$r \doteq 1.878 \tag{4.249}$$

and

$$\frac{r}{K} \doteq 5.459. \tag{4.250}$$

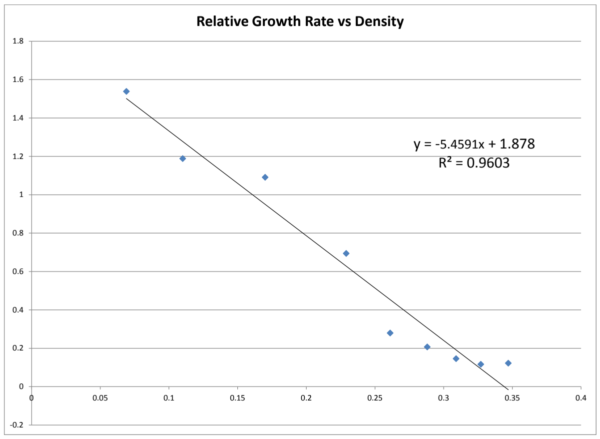

Figure 4.10.18: *Staphylococcus aureus* Linear Fit for the data in Table 4.6

Combining (4.249) and (4.250) yields the following estimate for the carrying capacity:

$$K \doteq 0.344. \tag{4.251}$$

The estimate for the carrying capacity in (4.251) is barely lower than the highest density value in Table 4.3, which is 0.347; thus, it is not a good estimate for the carrying capacity. In order to get a better estimate for K, consider Table 4.7, whose second column lists the average growth rates, not the *per–capita* growth rates, computed from the values in Table 4.3. Next, recall from the qualitative analysis of the Logistic equation in Section 3.1 that the highest rate of change for N occurs at the inflection point on the graph of the logistic curve; namely, when $N = K/2$, half of the carrying capacity. This observation allows us to obtain a better estimate for K by setting

$$\frac{K}{2} \doteq \frac{0.170 + 0.229}{2}, \tag{4.252}$$

the average of the two values of the optical density at which the top two values of the rate of change occur. We than obtain from (4.252) the estimate

$$K \doteq 0.399. \tag{4.253}$$

Adopting the estimate for K in (4.253), we now proceed to estimate the remaining parameters, N_o and r, in the formula for $N(t)$ in (4.248). In order to do this, we first rewrite the equation (4.248) in the form

$$\frac{K - N}{N} = \frac{K - N_o}{N_o} e^{-rt}. \tag{4.254}$$

Concentration (OD_{650})	Average Growth Rate
0.032	
0.039	0.014
0.069	0.060
0.110	0.082
0.170	0.120
0.229	0.118
0.261	0.064
0.288	0.054
0.309	0.042
0.327	0.036
0.347	0.040

Table 4.7: *Staphylococcus aureus* Rate of Growth Data

Next, take the natural logarithm on both sides of the equation in (4.254) to obtain

$$\ln\left(\frac{K-N}{N}\right) = \ln\left(\frac{K-N_o}{N_o}\right) - rt. \tag{4.255}$$

Thus, according to (4.255), plotting the values of $\ln[(K-N)/N]$ versus t, from the Table 4.3, should yield points that line up along a straight line of slope $-r$ and y–intercept $\ln[(K-N_o)/N_o]$. Using the estimate for K in (4.253) we may use the values for the slope and y–intercept of the least–squares linear fit to the data in order to estimate r and N_o. Table 4.8 was obtained from the values in

Time (hours)	$\ln[(K-N)/N]$
0.0	2.440
0.5	2.223
1.0	1.565
1.5	0.966
2.0	0.298
2.5	−0.298
3.0	−0.637
3.5	−0.953
4.0	−1.234
4.5	−1.513
5.0	−1.898

Table 4.8: *Staphylococcus aureus* Growth Data for Logistic Fit

Table 4.3 on page 46. Figure 4.10.19 shows a plot of the data in Table 4.8 along

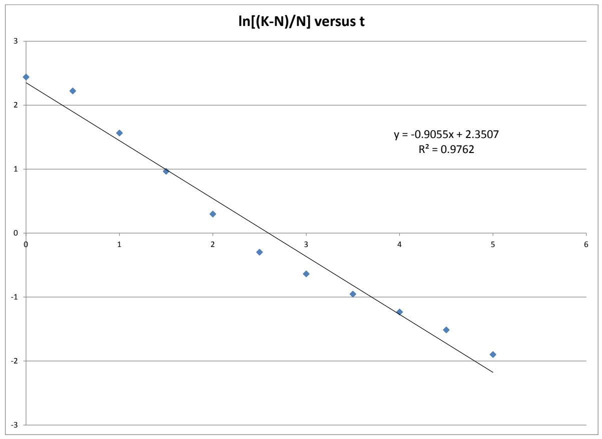

Figure 4.10.19: *Staphylococcus aureus* Linear Fit for the data in Table 4.8

with the least–squares linear fit, whose equation is

$$y = 2.3507 - 0.9055t. \tag{4.256}$$

Comparing equation (4.255) and (4.256), we obtain the estimates

$$r \doteq 0.906, \tag{4.257}$$

and

$$\ln\left(\frac{K - N_o}{N_o}\right) \doteq 2.351. \tag{4.258}$$

Combining (4.258) with (4.253) we get the following estimate for N_o:

$$N_o \doteq 0.035. \tag{4.259}$$

Using the estimated values for the parameters N_o, r and K in (4.259), (4.257) and (4.253), respectively, we obtain the following predicted values for the optical density based on the data in Table 4.3:

$$P(t) = \frac{0.014}{0.035 + 0.364e^{-0.906t}}, \quad \text{for } t \in \mathbb{R}. \tag{4.260}$$

Table shows the data from Table 4.3 along with the concentration values predicted with the formula in (4.260). Figure 4.10.20 shows a plot of the data in Table 4.3 along with the logistic model in (4.260) that we just fit to the data. We see that the Logistic model fits the data better then the Malthusian model demonstrated by comparison of the plots in Figures 4.7.14 and 4.10.20.

Time (hours)	Concentration (OD_{650})	Predicted Concentrations
0.0	0.032	0.035
0.5	0.039	0.053
1.0	0.069	0.077
1.5	0.110	0.109
2.0	0.170	0.148
2.5	0.229	0.192
3.0	0.261	0.237
3.5	0.288	0.278
4.0	0.309	0.313
4.5	0.327	0.340
5.0	0.347	0.360

Table 4.9: *Staphylococcus aureus* Growth Data and Predicted Values

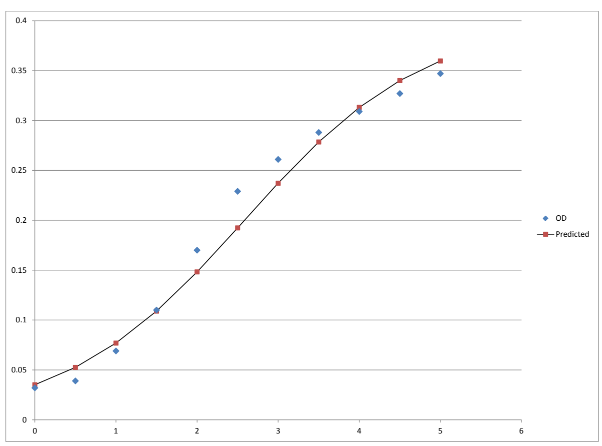

Figure 4.10.20: *Staphylococcus aureus* Logistic Fit for data in Table 4.3

Chapter 5

Applications of Differential Calculus: Part II

In the previous chapter we saw how separation of variables and a few integration techniques can be used to solve some differential equations that come up in modeling certain biological and physical phenomena. The models that we have analyzed so far were based on simplifying assumptions that made their mathematical analysis more tractable. In a lot of situations, though, simplifying assumptions cannot be made and the resulting models are consequently more complex and more difficult to analyze. In most situations, computing a formula for the solution to an initial value problem is neither feasible nor desirable; so, we need to resort to qualitative techniques. In Sections 3.1 and 4.8.2, we introduced a few of the ideas from qualitative analysis in context of the Logistic equation and linear first order equations with constant coefficients, respectively. In this chapter, we extend the ideas presented in those sections to more general models of the form

$$\frac{dN}{dt} = f(N), \tag{5.1}$$

where $f\colon I \to \mathbb{R}$ is a differentiable real valued function defined on some open interval, I, of real numbers. An example of an equation of the type in (5.1) is the following model for bacterial growth postulated by Monod in the 1930s and 1940s:

$$\frac{dN}{dt} = rN\left[\frac{C_o - N}{\gamma a + C_o + N}\right], \tag{5.2}$$

for some parameters r, C_o, γ and a. It is tempting to use separation of variables to solve the equation in (5.2). Perhaps the partial fractions might help to evaluate the integral involving N. It might even be possible to solve for N as a function of t, given some initial condition, $N(0) = N_o$. However, as demonstrated in Section 3.1 for the Logistic equation. qualitative information obtained without solving the equation might be sufficient to obtain a very good idea of what the model is predicting. We will demonstrate that in this chapter for the case of the equation in (5.2) and the general equation (5.1).

In Section 4.8.2 we introduced the concept of an equilibrium solution. We say that $\overline{N}$ is an equilibrium solution of (5.1) if $\overline{N}$ is a solution to the equation

$$f(N) = 0;$$

For example, the equation in (5.2) has two equilibrium solutions:

$$\overline{N}_1 = 0 \quad \text{and} \quad \overline{N}_2 = C_o.$$

As in the analysis presented for linear coefficients, we would like to learn whether an equilibrium solution is stable or not. We saw in Section 4.8.2 that the stability of the equilibrium point $\overline{y} = \dfrac{b}{a}$ for the linear equation

$$\frac{dy}{dt} = ay + b,$$

for $a \neq 0$, is determined by the sign of a. In this chapter we will learn that the stability of an equilibrium point, $\overline{y}$, for the first order differential equation

$$\frac{dy}{dt} = f(y),$$

for the case in which $f'(\overline{y}) \neq 0$ is determined by the sign of $f'(\overline{y})$. To obtain this result, we will use the linear approximation to the the function f around $\overline{y}$ provided by the first derivative:

$$f(y) = f(\overline{y}) + f'(\overline{y})(y - \overline{y}) + E(\overline{y}, y),$$

where $E(\overline{y}, y)$ measures the error in approximating f around $\overline{y}$ by the linear approximation

$$L(y) = f(\overline{y}) + f'(\overline{y})(y - \overline{y}).$$

5.1 Linear Approximations

Let $f\colon I \to \mathbb{R}$ be a differentiable function over an open interval containing a.

Definition 5.1.1 (Linear Approximation)**.** The function given by

$$L(x; a) = f(a) + f'(a)(x - a), \quad \text{for all } x \in \mathbb{R}, \tag{5.3}$$

is called the linear approximation for f around a. Note that

$$y = L(x; a)$$

gives the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. For this reason, the expression in (5.3) is also known as the tangent line approximation around a of the function f; in some texts, $L(x; a)$ is also referred to as the local linearization of f at a.

Example 5.1.2. Let $f\colon (0,\infty) \to \mathbb{R}$ be given by

$$f(x) = \ln x, \qquad \text{for all } x > 0.$$

Give the linear approximation to f near $a = 1$.
Solution: Compute

$$L(x;1) = f(1) + f'(1)(x-1), \qquad \text{for all } x \in \mathbb{R},$$

where $f'(x) = \dfrac{1}{x}$ for $x > 0$, so that $f'(1) =$ and

$$L(x;1) = x - 1, \qquad \text{for } x \in \mathbb{R}.$$

Figure 5.1.1 shows the graph of $y = f(x)$ and the linear approximation to f around $a = 1$. □

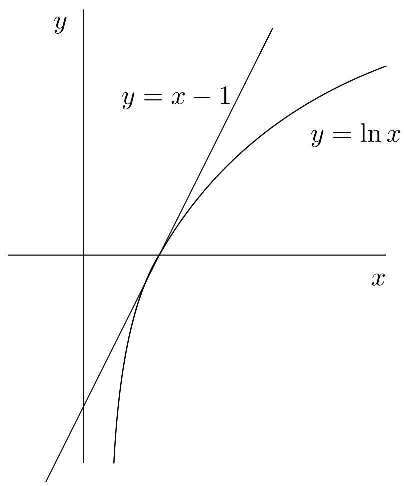

Figure 5.1.1: Sketch of graph of $y = \ln(x)$ and Linear Approximation at $a = 1$

Definition 5.1.3 (Error Term in Linear Approximation)**.** Let $f\colon I \to \mathbb{R}$ be a differentiable function over an open interval containing a, and denote by $L(x;a)$ its linear approximation. Define

$$E(a;x) = f(x) - L(x;a), \qquad \text{for } x \in I. \tag{5.4}$$

We then have that

$$f(x) = f(a) + f'(a)(x-a) + E(a;x), \qquad \text{for } x \in I, \tag{5.5}$$

where

$$\lim_{x\to a} \frac{E(a;x)}{x-a} = 0. \tag{5.6}$$

Remark 5.1.4. The expression in (5.6) follows from (5.4), (5.5), and the definition of the derivative of f at a; namely,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \tag{5.7}$$

In fact, rewriting (5.5) we have that, for $x \neq a$,

$$\frac{E(a;x)}{x-a} = \frac{f(x) - f(a)}{x - a} - f'(a); \tag{5.8}$$

thus, taking the limit as $x \to a$ on both sides of (5.8) and using (5.7), we obtain that

$$\lim_{x\to a} \frac{E(a;x)}{x-a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - f'(a) = f'(a) - f'(a) = 0,$$

which is (5.6).

Next, we derive a formula for computing $E(a; x)$ in (5.4) which will allow us to obtain useful estimates.

Proposition 5.1.5 (Estimating the Error)**.** Assume that f is a twice differentiable function defined on an interval, I, that contains a. Then, the error term in (5.4) is given by

$$E(a;x) = \int_a^x f''(t)(x-t)\ \mathrm{d}t, \qquad \text{for } x \in I. \tag{5.9}$$

Hence, if $|f''(x)| \leq M$ for all $x \in I$ and some constant M, then

$$|E(x;a)| \leq \frac{M}{2}|x-a|^2, \qquad \text{for } x \in I. \tag{5.10}$$

Proof: Use integration by parts to evaluate the integral on the right–hand side of (5.9) by setting

$$\begin{array}{rcl} u = x - t & \text{and} & dv = f''(t)\ dt \\ \text{then, } du = -\ dt & \text{and} & v = f'(t), \end{array}$$

so that

$$\int_a^x f''(t)(x-t)\ \mathrm{d}t \quad = \quad f'(t)(x-t)\Big|_a^x + \int_a^x f'(t)\ dt \tag{5.11}$$

from which we get that

$$\int_a^x f''(t)(x-t)\ \mathrm{d}t \quad = \quad -f'(a)(x-a) + f(x) - f(a), \tag{5.12}$$

where we have used the Fundamental Theorem of Calculus in evaluation the right–most integral in (5.11). The equation in (5.12) can be rewritten as

$$\int_a^x f''(t)(x-t)\ \mathrm{d}t \quad = \quad f(x) - L(x;a), \tag{5.13}$$

by the definition of $L(x;a)$ in (5.3). Comparing the equations in (5.4) and (**??**) yields (5.9).

To complete the proof of Proposition 5.1.5, assume that

$$|f''(x)| \leq M, \quad \text{for all } x \in I, \tag{5.14}$$

and some constant M. Take absolute values on both sides of (5.9) to obtain the estimate

$$|E(a;x)| \leqslant \int_a^x |f''(t)||x-t| \, \mathrm{d}t, \quad \text{for } x \in I \text{ with } x > a. \tag{5.15}$$

Next, use the estimate for $|f''(t)|$ in (5.14) in the integral in (5.15) and integrate to obtain from (5.15) that

$$|E(a;x)| \leqslant \frac{M}{2}|x-a|^2, \quad \text{for } x \in I \text{ with } x > a. \tag{5.16}$$

Similar calculations show that

$$|E(a;x)| \leqslant \frac{M}{2}|x-a|^2, \quad \text{for } x \in I \text{ with } x < a. \tag{5.17}$$

Combining (5.16) and (5.19) gives the estimate in (5.10). ■

Example 5.1.6. Estimate the cosine of 1.
Solution: Here, 1 represents the measure of an angle in radians. Observe that

$$\frac{\pi}{3} \approx 1.0472;$$

Thus, we can estimate cos(1) by using the linear approximation to $f(x) = \cos x$ at $a = \pi/3$; namely,

$$L(\pi/3) = f(\pi/3) + f'(\pi/3)\left(x - \frac{\pi}{3}\right), \quad \text{for } x \in \mathbb{R},$$

where

$$f'(x) = -\sin x, \quad \text{for } x \in \mathbb{R}.$$

Thus,

$$L(x;\pi/3) = \cos(\pi/3) - \sin(\pi/3)\left(x - \frac{\pi}{3}\right), \quad \text{for } x \in \mathbb{R},$$

or

$$L(x;\pi/3) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right), \quad \text{for } x \in \mathbb{R}. \tag{5.18}$$

Thus, using the linear approximation to $\cos x$ at $a = \pi/3$ in (5.19), we get that

$$\cos(1) \approx L(1;\pi/3) = \frac{1}{2} - \frac{\sqrt{3}}{2}(1 - 1.0472),$$

so that

$$\cos(1) \approx 0.540876. \tag{5.19}$$

Using the estimate for the error in the approximation in (5.10), with $M = 1$ since $|\cos''(t)| = |cost| \leqslant 1$ for all $t \in \mathbb{R}$, we see that

$$|E(1;\pi/3)| \leqslant \frac{1}{2}|1 - 1.0472|^2 \doteq 0.001.$$

Thus, the estimate in (5.19) is accurate to two decimal places, so that

$$\cos(1) \doteq 0.54.$$

□

5.2 The Principle of Linearized Stability

Let $f\colon I \to \mathbb{R}$ be a differentiable function with continuous derivative defined on some interval I. In this section we present a qualitative analysis of the differential equation

$$\frac{dy}{dt} = f(y) \tag{5.20}$$

based on linear approximations of f around its equilibrium points. We will be dealing only with isolated equilibrium points.

Definition 5.2.1 (Isolated Equilibrium Point)**.** A point $\overline{y} \in I$ is said to be an equilibrium point of the differential equation in (5.20) if $\overline{y}$ solves the equation

$$f(y) = 0. \tag{5.21}$$

If $\overline{y}$ is the only solution to (5.21) in some open interval containing $\overline{y}$, then $\overline{y}$ is said to be isolated.

Example 5.2.2. The points $\overline{y}_1 = 0$ and $\overline{y}_2 = 1$ are isolated equilibrium points of the differential equation

$$\frac{dy}{dt} = y - y^2 \tag{5.22}$$

To see why this assertion is true, observe that the interval $(-1/2, 1/2)$ contains only one equilibrium point, $\overline{y}_1 = 0$, while the interval $(1/2, 3/2)$ contains only one equilibrium point, $\overline{y}_2 = 1$.

Let $\overline{y}$ denote an isolated equilibrium point of the equation in (5.20) and $y_o \neq \overline{y}$ represent any point in an open interval around $\overline{y}$ which contains no equilibrium points other than $\overline{y}$. We consider the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(y); \\ y(0) = y_o. \end{cases} \tag{5.23}$$

We are interested in the behavior of solution, $y = y(t)$, to the initial value problem as time increases. Two things can happen: (i) either $y(t)$ will tend towards $\overline{y}$, or (ii) $y(t)$ will tend away from $\overline{y}$. In the first case we say that the equilibrium solution, $\overline{y}$, is stable, while is case (ii) we say that $\overline{y}$ unstable. We will make these notions more precise in the following definitions.

Definition 5.2.3 (Stability)**.** Let $\overline{y}$ be an isolated equilibrium point of the equation (5.20). The equilibrium solution $y(t) = \overline{y}$, for all $t \in \mathbb{R}$, is said to be **stable** if and only if

(i) There exists an interval J contained in I such that, if $y_o \in J$, then the initial value problem (5.23) has a solution $y = y(t)$ that exists for all $t > 0$, and

(ii) the solution $y(t)$ obtained in part (i) tends towards $\overline{y}$ as t increases.

Definition 5.2.4 (Asymptotic Stability)**.** Let $\overline{y}$ be an isolated equilibrium point of the equation (5.20). The equilibrium solution $y(t) = \overline{y}$ for all $t \in \mathbb{R}$ is said to be **asymptotically stable** if and only if it is stable and, in addition to (i) and (ii) in Definition 5.2.3, the solution $y(t)$ obtained in part (i) of Definition 5.2.3 satisfies

$$\lim_{t\to\infty} y(t) = \overline{y}.$$

Definition 5.2.5 (Unstable equilibrium point)**.** Let $\overline{y}$ be an isolated equilibrium point of the equation (5.20). The equilibrium solution $y(t) = \overline{y}$ for all $t \in \mathbb{R}$ is said to be **unstable** if and only if it is not stable.

Example 5.2.6. The initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y - y^2; \\ y(0) = y_o, \end{cases} \tag{5.24}$$

has the solution

$$y(t) = \frac{y_o}{y_o + (1 - y_o)e^{-t}}, \tag{5.25}$$

which can be obtained by separating variables and using partial fractions to evaluate the integral involving y. Observe that for $y_o > 0$, the function y given by (5.25) is defined for all values of $t > 0$. To see why this is the case, note that, if $0 < y_o \leqslant 1$, then

$$y_o + (1 - y_o)e^{-t} \geqslant y_o > 0, \quad \text{for all } t \in \mathbb{R}.$$

Thus, the denominator of the expression defining $y(t)$ in (5.25) is nonzero for all $t \in \mathbb{R}$, and therefore $y(t)$ is defined for all $t \in \mathbb{R}$. For the case in which $y_o > 1$, we have that

$$y_o - 1 > 0,$$

so that

$$(y_o - 1)e^{-t} \leqslant y_o - 1, \quad \text{for } t \geqslant 0, \tag{5.26}$$

since $e^{-t} \leqslant 1$ for $t \geqslant 0$. Multiplying on both sides of the inequality in (5.26) by -1 then yields

$$(1 - y_o)e^{-t} \geqslant 1 - y_o, \quad \text{for } t \geqslant 0,$$

so that

$$y_o + (1 - y_o)e^{-t} \geqslant 1 > 0, \quad \text{for } t \geqslant 0,$$

and therefore the denominator of the expression defining $y(t)$ in (5.25) is nonzero for $t \geqslant 0$, and therefore $y(t)$ is defined for $t \geqslant 0$.

It follows from the definition of $y(t)$ in (5.25) that, for $y_o > 0$,

$$\lim_{t\to\infty} y(t) = 1.$$

Thus, $\overline{y}_2 = 1$ is an asymptotically stable equilibrium point of the equation (5.22) in Example 5.2.2, according to Definition 5.2.4.

We next consider equilibrium point $\overline{y}_1 = 0$. Differentiating with respect to t the expression for $y(t)$ in (5.25), we obtain

$$y'(t) = \frac{y_o(1-y_o)e^{-t}}{(y_o + (1-y_o)e^{-t})^2}. \tag{5.27}$$

Thus, for any $y_o > 0$ close to 0, $y'(t) > 0$ so that $y(t)$ increases from y_o as t increases, and so $y(t)$ tends away from $\overline{y}_1 = 0$. Thus, it follows from Definition 5.2.5 that $\overline{y}_1 = 0$ is unstable.

In Example 5.2.6 we we able to write down a formula for computing a solution to the differential equation subject to the initial condition. The formula allowed us to verify the conditions in the definition of asymptotic stability for the equilibrium point $\overline{y}_2 = 1$. We also used the formula for $y(t)$ and its derivative to show that $\overline{y}_1 = 0$ is unstable. In the general case of equation (5.20) we might not be able to obtain a formula for the solution to the initial value problem (5.23). Thus, if all we know for sure about f is that it is differentiable and has continuous derivative on some interval of real values, we need to resort to the general theory of ordinary differential equations in order to answer any question regarding stability of equilibrium points. We state here a few of the results that may be found in several texts in differential equations; for example, see [BC87].

Theorem 5.2.7 (Local Existence and Uniqueness)**.** Suppose that f and f' are continuous in an interval which contains y_o, then the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(y); \\ y(t_o) = y_o, \end{cases} \tag{5.28}$$

has one, and only one, solution defined in some interval around t_o.

The reason Theorem 5.2.7 is called the Local Existence and Uniqueness Theorem is that a unique solution to the initial value problem (5.28) is guaranteed to exist in some interval around t_o. Further developments of the theory of ordinary differential equations show that the solution exist in some maximal interval containing t_o.

Example 5.2.8. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y^2; \\ y(0) = y_o, \end{cases} \tag{5.29}$$

for $y_o > 0$.
Solution: Separating variables we obtain that

$$\int \frac{1}{y^2}\, dy = \int dt,$$

which integrates to

$$-\frac{1}{y} = t + c_1, \tag{5.30}$$

for arbitrary constant c_1. Multiplying both sides of the equation in (5.30) by -1 and setting $c = -c_1$, we obtain from (5.30) that

$$\frac{1}{y} = c - t. \tag{5.31}$$

Solving for y in (5.31) yields the general solution

$$y(t) = \frac{1}{c - t}. \tag{5.32}$$

Next, use the initial condition $y(0) = y_o$ to obtain from (5.32) that

$$\frac{1}{c} = y_o,$$

from which we get that $c = 1/y_o$, so that

$$y(t) = \frac{1}{1/y_o - t},$$

or

$$y(t) = \frac{y_o}{1 - y_o t}. \tag{5.33}$$

Note that the function given in (5.33) is differentiable for $t < 1/y_o$. Thus, the maximal interval of existence for the initial value problem in (5.29) is $(-\infty, 1/y_o)$. □

Remark 5.2.9. The result of Example 5.2.8 shows that the equilibrium point, $\overline{y} = 0$ for the differential equation

$$\frac{dy}{dt} = y^2$$

is unstable since condition (i) in Definition 5.2.3 is not satisfied; in other words, the solution $y(t)$ to the initial value problem in (5.29) does not exist for all $t > 0$.

In some situations we are able to tell, without solving the equation, whether a solutions exists for all time t (or at least for $t > 0$). Here is an example of a result that yields existence for all time, t, and which may be found in [BC87].

Theorem 5.2.10 (Global existence and long–term behavior)**.** Suppose that f and f' are continuous in some interval. Let $\overline{y}_1$ and $\overline{y}_2$ be two consecutive, isolated equilibrium points of the differential equation in (5.20) with $\overline{y}_1 < \overline{y}_2$. Then, for any y_o such that $\overline{y}_1 < y_o < \overline{y}_2$, the solution, $y(t)$, to the initial value problem in (5.23) exists for all $t \geq 0$. Furthermore, $\lim_{t\to\infty} y(t)$ exists and it equals to one of $\overline{y}_1$ or $\overline{y}_2$, depending on the stability properties of $\overline{y}_1$ and $\overline{y}_2$. Likewise, $y(t)$ exists for all $t \leq 0$ and it approaches an equilibrium solution as $t \to -\infty$.

Example 5.2.11. The differential equation

$$\frac{dy}{dt} = y - y^2$$

has equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = 1$; so that $\overline{y}_1 < \overline{y}_2$, as required by Theorem 5.2.10. It then follows from the theorem that, for any y_o with $0 < y_o < 1$, the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y - y^2; \\ y(0) = y_o, \end{cases} \tag{5.34}$$

has a solutions that exists for all values of $t \in \mathbb{R}$. This was demonstrated in Example 5.2.6. We also so in Example 5.2.6 that, for $y_o > 0$, the solution, $y = y(t)$, for the initial value problem 5.34 satisfies

$$\lim_{t\to\infty} y(t) = 1.$$

It can also be verified that, for $0 < y_o < 1$, the solution, $y = y(t)$, for the initial value problem 5.34 satisfies

$$\lim_{t\to-\infty} y(t) = 0.$$

The final result from the theory of ordinary differential equations that we will quote in this section will allow us to tell whether $\overline{y}$ is a stable or unstable equilibrium point of the equation in (5.20) by looking at the sign of $f'(\overline{y})$ for the case in which $f'(\overline{y}) \neq 0$. It is called the principle of linearized stability. When it is applicable, it is a very powerful result.

We begin by defining the linearization of the differential equation (5.20) around an isolated equilibrium point, $\overline{y}$.

Let $y = y(t)$ denote a solution to the differential equation in (5.20) and put

$$u = y - \overline{y}. \tag{5.35}$$

Differentiating with respect to t on both sides of (5.35) yields

$$\frac{du}{dt} = \frac{dy}{dt} = f(y), \tag{5.36}$$

where we have used the assumption that y solves the differential equation in (5.20). Next, use the linear approximation to f at $\overline{y}$ to rewrite (5.36) as

$$\frac{du}{dt} = f(\overline{y}) + f'(\overline{y})(y - \overline{y}) + E(y, \overline{y}), \tag{5.37}$$

where $E(y, \overline{y})$ is the error in the linear approximation. We now use the assumption that $\overline{y}$ is an equilibrium point and the definition of u in (5.35) to get from (5.37) that

$$\frac{du}{dt} = f'(\overline{y})u + E(y, \overline{y}). \tag{5.38}$$

When y is very close to $\overline{y}$, we obtain from (5.38) that the equation in (5.20) can be approximated by the equation linear differential equation

$$\frac{du}{dt} = f'(\overline{y})u. \tag{5.39}$$

The equation in (5.39) is called the linearization of the equation in (5.20).

Observe that, for $f'(\overline{y}) \neq 0$, the linear equation in (5.39) has one equilibrium point, namely $\overline{u} = 0$. Observe also that the linearized equation (5.39) can be solved to yield

$$u(t) = u_o e^{f'(\overline{y})t}, \quad \text{for all } t \in \mathbb{R}, \tag{5.40}$$

where $u_o = u(0)$. Thus, if $f'(\overline{y}) < 0$, we get form (5.40) that

$$\lim_{t\to\infty} u(t) = 0;$$

so that, if $f'(\overline{y}) < 0$, then $\overline{u} = 0$ is an asymptotically stable equilibrium point for the linearized equation (5.39). On the other hand, if $f'(\overline{y}) > 0$, then, for any $u_o \neq 0$, we see from (5.40) that $u(t)$ tends away from 0 as t increases. Hence, if $f'(\overline{y}) > 0$, then $\overline{u} = 0$ is unstable. The principle of linearized stability states that, if $f'(\overline{y}) \neq 0$, then the isolated equilibrium $\overline{y}$ of the equation in (5.20) inherits the same kind of stability property as $\overline{u} = 0$ for the linearized equation (5.39). In other words, if $f'(\overline{y}) < 0$, then $\overline{y}$ is an asymptotically stable; and, if $f'(\overline{y}) > 0$, then $\overline{y}$ is unstable.

Theorem 5.2.12 (Principle of linearized stability)**.** Suppose that f and f' are continuous in some open interval containing an isolated equilibrium point, $\overline{y}$, of the differential equation (5.20). If $\overline{u} = 0$ is asymptotically stable for the linearized equation (5.39), then $\overline{y}$ is also asymptotically stable for the non–linear equation (5.20); if $\overline{u} = 0$ is unstable for (5.39), then it is also unstable for (5.20). In other words, if $f'(\overline{y}) < 0$, then $\overline{y}$ is asymptotically stable, and if $f'(\overline{y}) > 0$, then $\overline{y}$ is unstable.

Example 5.2.13. For the equation in Example 5.2.6, $f(y) = y - y^2$, so that $f'(y) = 1 - 2y$. We then have that

$$f'(0) = 1 > 0,$$

so that $\overline{y}_1 = 0$ is unstable, by the principle of linearized stability. On the other hand, since

$$f'(1) = -1 < 0,$$

and therefore $\overline{y}_2 = 1$ is asymptotically stable, by the principle of linearized stability. Both of these statements were shown to be true in Example 5.2.6 by verifying the conditions in Definitions 5.2.3, 5.2.4 and 5.2.5.

Example 5.2.14. Consider the Monod model for bacterial growth in (5.2); namely,

$$\frac{dN}{dt} = rN\left[\frac{C_o - N}{\gamma a + C_o + N}\right], \tag{5.41}$$

for positive parameters r, C_o, γ and a.

Solution: Set $f(N) = rN\left[\dfrac{C_o - N}{\gamma a + C_o + N}\right]$. Then, $\overline{N}_1 = 0$ and $\overline{N}_2 = C_o$ are equilibrium points of the equation in (5.41). In order to determine the nature of the stability of $\overline{N}_1$ and $\overline{N}_2$, we apply the principle of linearized stability, Theorem 5.2.12. Thus, compute

$$f'(N) = r\frac{(\gamma a + C_o + N)(C_o - 2N) - N(C_o - N)}{(\gamma a + C_o + N)^2}, \qquad \text{for } N \geqslant 0. \tag{5.42}$$

It follows from (5.42) that

$$f'(0) = \frac{rC_o}{\gamma a + C_o} > 0,$$

so that $\overline{N}_1 = 0$ is unstable, by the principle of linearized stability. Similarly, we compute using (5.42) that

$$f'(C_o) = -\frac{rC_o}{\gamma a + 2C_o} < 0,$$

which shows that $\overline{N}_2 = C_o$ is asymptotically stable, by the principle of linearized stability. □

Remark 5.2.15. The results in Example 5.2.14 in conjunction with global existence and long–term behavior theorem (Theorem 5.2.10), allow us to infer from the Monod model for bacterial growth in (5.41) that, for initial population densities, $N(0) = N_o$, such that $0 < N_o < C_o$, the population density will tend to the limiting value of $\overline{N}_2 = C_o$ as $t \to \infty$.

Remark 5.2.16. Examples 5.2.13 and 5.2.14 illustrate the fact that the principle of linearized stability, when applicable, is indeed very powerful since its application gives as a very good picture of the long–term behavior of solutions to a differential equation without having to solve the differential equation. However, the principle is not always applicable. For instance, consider the differential equation

$$\frac{dy}{dt} = y^2. \tag{5.43}$$

In this case $f(y) = y^2$ and $\overline{y} = 0$ is the only equilibrium point. However. $f'(y) = 2y$, so that $f'(0) = 0$ and, therefore, the principle of linearized stability does not apply. Thus, in order to determine the stability properties of $\overline{y} = 0$, other means of analysis have to be employed. In Examples 5.2.8 and 5.2.9, we showed that $\overline{y} = 0$ is an unstable equilibrium solution of the equation (5.43).

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