Solutions to Assignment #10

1. For real numbers a and b with $a < b$, let (a, b) denote the open interval from a to b:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

A subset, D, of the real numbers is said to be **dense** in $\mathbb{R}$ if and only if for every open interval, (a, b),

$$(a, b) \cap D \neq \emptyset;$$

that is, the intersection of any open interval with D is nonempty.

Use the fact that between any two distinct real numbers there exists a rational number to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$ according to the definition given above.

***Solution**:* We show that $(a, b) \cap \mathbb{Q} \neq \emptyset$ for any, nonempty open interval (a, b).

Proof: Since (a, b) is not empty, $a < b$. Next, use the fact that between any two distinct real numbers there exists a rational number to get $q \in \mathbb{Q}$ such that

$$a < q < b.$$

Thus, $q \in (a, b)$ and therefore $q \in (a, b) \cap \mathbb{Q}$. Hence, $(a, b) \cap \mathbb{Q}$ is not empty. □

□

2. Show that $\mathbb{Z}$ is not dense in $\mathbb{R}$.

***Solution**:* Observe that the interval $(0, 1)$ contains no integers. For if $m \in \mathbb{Z}$ and $m \in (0, 1)$ then $m > 0$ and $m < 1$. However, $m \geqslant 1$, since $m \in \mathbb{N}$. We have therefore arrived at a contradiction. Thus, $(0, 1) \cap \mathbb{Z} = \emptyset$ and therefore $\mathbb{Z}$ cannot be dense in $\mathbb{R}$. □

3. Let $a, b \in \mathbb{R}$ with $a < b$. Prove that the set $(a, b) \cap \mathbb{Q}$ is infinite.

Proof: Assume by way of contradiction that $(a, b) \cap \mathbb{Q}$ is finite. Then,

$$(a, b) \cap \mathbb{Q} = \{q_1, q_2, \ldots, q_n\}, \tag{1}$$

where the rational numbers $q_1, q_2, \ldots, q_n$ may be ordered as follows:

$$a < q_1 < q_2 < \cdots < q_n < b. \tag{2}$$

Since there is a rational number, q, such that

$$q_n < q < b, \tag{3}$$

it follows from (2) that $q \in (a, b)$. Thus, $q \in (a, b) \cap \mathbb{Q}$. However, q is not listed in the definition (1) since $q > q_i$ for $i = 1, 2, \ldots, n$, by the inequalities in (2) and (3). We have therefore arrived at a contradiction. Consequently, $(a, b) \cap \mathbb{Q}$ is infinite. $\square$

4. Given sets A and B, the set of elements in A which are not in B is denoted by $A \setminus B$; that is,

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Thus, for instance, the set $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

Prove that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: We need to show that $(a, b) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ for any nonempty open interval (a, b).

Let (a, b) be a nonempty open interval of real numbers. Then,

$$a < b.$$

By the result of Problem 5 in Assignment #9, there exists and irrational number, α, between a and b. Thus, $\alpha \in (a, b)$ and therefore

$$\alpha \in (a, b) \cap (\mathbb{R} \setminus \mathbb{Q}),$$

which shows that $(a, b) \cap (\mathbb{R} \setminus \mathbb{Q})$ is not empty. Hence, $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. $\square$

5. Let $q \in \mathbb{Q}$ and α be an irrational number. Prove that

   (a) if $q \neq 0$, then $q\alpha$ is irrational.

   Proof: Let $q \in \mathbb{Q}$, $q \neq 0$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume, by way of contradiction, that $q\alpha$ is rational. It then follows that $q^{-1}(q\alpha) \in \mathbb{Q}$ since $\mathbb{Q}$ is a field. Consequently, $\alpha \in \mathbb{Q}$, which is a contradiction. Therefore, $q\alpha$ is irrational, if $q \in \mathbb{Q}$ and $q \neq 0$. $\square$

   (b) $q + \alpha$ is irrational for all $q \in \mathbb{Q}$.

Proof: Let $q \in \mathbb{Q}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume, by way of contradiction, that $q + \alpha$ is rational. It then follows that $(q + \alpha) - q \in \mathbb{Q}$ since $\mathbb{Q}$ is a field. Consequently, $\alpha \in \mathbb{Q}$, which is a contradiction. Therefore, $q + \alpha$ is irrational. $\square$

(c) What can you say about α^q? ***Answer:*** α^q could be rational or irrational. For instance, if $\alpha = \sqrt{2}$ and $q = 1$, then $\alpha^q = \sqrt{2}$, which is irrational.
On the other hand, if if $\alpha = \sqrt{2}$ and $q = 2$, then $\alpha^q = 2$, which is rational. $\square$