Solutions to Assignment #2

1. Let P, Q and R denote propositions. Use a truth–table to verify that the implication $P \Rightarrow (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \Rightarrow R$.

    Solution:

P	Q	R	$\neg Q$	$Q \vee R$	$P \wedge \neg Q$	$P \Rightarrow (Q \vee R)$	$(P \wedge \neg Q) \Rightarrow R$
T	T	T	F	T	F	T	T
T	T	F	F	T	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	F	F	T	T

    Since the truth–values in the last two columns of the table are the same, it follows that

    $$[P \Rightarrow (Q \vee R)] \iff [(P \wedge \neg Q) \Rightarrow R].$$

    $\square$

2. Let m and n denote integers. Prove that if 2 divides mn, then either 2 divides m or 2 divides n.

    Suggestion: Use the result of the previous problem and prove the implication: If 2 divides mn and 2 does not divide m, then 2 divides n.

    Solution: By the result of the previous problem, it suffices to prove the implication

    If 2 divides mn and 2 does not divide m, then 2 divides n.

    Proof: Let m and n denote integers and assume that 2 divides mn and 2 does not divide m. It then follows that

    $$mn = 2k \quad \text{and} \quad m = 2\ell + 1$$

    for some integers k and ℓ. Substituting $2\ell + 1$ for m in the equation $mn = 2k$, we then obtain that

    $$(2\ell + 1)n = 2k,$$

    from which we obtain that

    $$2\ell n + n = 2k. \tag{1}$$

Solving for n in (1) yields
$$n = 2k - 2\ell n,$$

or
$$n = 2(k - \ell n),$$

which shows that 2 divides n. □

□

3. Use mathematical induction to prove that every non–empty subset of the natural numbers must have a smallest element.

Suggestion: Let A denote a non–empty subset of $\mathbb{N}$. We claim that A must have a smallest element. Argue by contradiction: Assume that A has no smallest element and let S denote the set of natural numbers that are not in A.

(a) Prove that $1 \in S$.

(b) Prove that $k \in S$ for all $k \in \{1, 2, \ldots, n\}$ implies that $n + 1 \in S$.

(c) Deduce that $S = \mathbb{N}$. Explain why this is a contradiction.

Proof: We argue by contradiction.

Assume that there is a non–empty subset, A, of the natural numbers which has no smallest element. Define S to be the complement of A in $\mathbb{N}$; that is,

$$S = \{n \in \mathbb{N} \mid n \notin A\}$$

We show the following about S:

(a) $1 \in S$.

To see why this is the case, observe that if $1 \notin S$, then $1 \in A$, by the definition of S, therefore 1 would be the smallest element of A. But, we are assuming that A has no smallest element. Thus, $1 \in S$ must be true.

(b) $k \in S$, for $k = 1, 2, 3 \ldots, n$, implies that $n + 1 \in S$.

To prove this assertion, we argue by contradiction. Assume that $k \in S$ for $k = 1, 2, \ldots, n$ and that $n + 1 \notin S$. It then follows that $n + 1 \in A$. Hence, since $k \notin A$ by assumption, it follows that $n + 1$ is the smallest element of A. However, this contradicts the assumption that A has no smallest element.

(c) By the results in (a) and (b) that we have just proved, it follows from principle of strong induction (see Theorem 4.6 on page 59 in Schramm's text) that $S = \mathbb{N}$. This then implies that $A = \emptyset$, which is a contradiction to the assumption that A is non–empty.

The contradiction in part (c) above allows us to conclude that every non–empty subset of $\mathbb{N}$ must have a smallest element. □

4. Find the smallest natural number that can be written as the sum of three prime numbers, but cannot be written as the sum of two prime numbers.

Answer: The number is 11, since $11 = 3 + 3 + 5$, but no two prime numbers add up two 11. In fact, the natural numbers smaller than or equal to 11 which can be obtained as sums of three primes are

$$
\begin{aligned}
2 + 2 + 2 &= 6 \\
2 + 2 + 3 &= 7 \\
2 + 3 + 3 &= 8 \\
2 + 2 + 5 &= 9 \\
3 + 3 + 3 &= 9 \\
2 + 3 + 5 &= 10 \\
2 + 2 + 7 &= 11 \\
2 + 7 + 2 &= 11 \\
3 + 3 + 5 &= 11.
\end{aligned}
$$

Out of these, only the number 11 is not on the list of numbers less than or equal to 11 that can be obtained as the sum of two prime numbers; namely,

$$
\begin{aligned}
2 + 2 &= 4 \\
3 + 3 &= 6 \\
2 + 5 &= 7 \\
3 + 5 &= 8 \\
2 + 7 &= 9 \\
3 + 7 &= 10 \\
5 + 5 &= 10.
\end{aligned}
$$

□

5. Let $m, n \in \mathbb{Z}$ with $0 < m < n$. Define $S = \{n - km \mid k \in \mathbb{Z} \text{ and } n - mk \geqslant 0\}$.

(a) Prove that S has a smallest element and call it r.

Proof: First, observe that S is non–empty since $n - m > 0$ so that $n - m \in S$. Thus, by the well–ordering principle, S has a smallest element, r, which could be 0. $\square$

(b) Prove that $r \in \{0, 1, \ldots m - 1\}$.

Suggestion: Show that $r \geqslant m$ is impossible.

Proof: Assume, by way of contradiction, that $r \geqslant m$. Since, $r \in S$, there exists $k \in \mathbb{Z}$ such that $r = n - km$. It then follows that $n - km \geqslant m$, from which we get that

$$n - (k + 1)m \geqslant 0.$$

Thus, $n - (k + 1)m \in S$. However,

$$n - (k + 1)m = n - km - m = r - m < r,$$

which contradicts the fact that r is the smallest element in S. Hence, r must lie in the set $\{0, 1, \ldots, m - 1\}$. $\square$

(c) Prove: Given positive integers, m and n, with $m < n$, there exist unique integers, q and r, such that,

$$n = qm + r \quad \text{where} \quad r \in \{0, 1, \ldots m - 1\}.$$

Note: This is a special case of the Division Algorithm.

Proof: Let r denote the smallest element of S. Then, r is unique and $r = n - qm$ for some integer q, where $0 \leqslant r \leqslant m - 1$. Then,

$$n = qm + r.$$

To show that q is unique, assume that there is some other integer, q_1, with the property that $n = q_1 m + r$. Then, subtracting this equation from $n = qm + r$,

$$0 = (q - q_1)m,$$

which implies that $q - q_1 = 0$ since $m > 0$. Thus, $q_1 = q$. $\square$