Solutions to Assignment #4

1. Let $a, b \in \mathbb{R}$. Prove that

$$a^2 + b^2 = 0 \text{ if and only if } a = 0 \text{ and } b = 0.$$

Proof: First observe that, since $0 \cdot x = 0$ for all $x \in \mathbb{R}$, it follows that $0^2 = 0$. Thus, if $a = 0$ and $b = 0$, then

$$a^2 + b^2 = 0 + 0 = 0.$$

Conversely, we prove that $a^2 + b^2 = 0$ implies that $a = 0$ and $b = 0$ by showing the contrapositive:

$$a \neq 0 \text{ or } b \neq 0 \Rightarrow a^2 + b^2 \neq 0.$$

Assume that $a \neq 0$. Then, $a^2 > 0$. Thus, adding b^2 on both sides,

$$a^2 + b^2 > 0 + b^2 = b^2 \geqslant 0,$$

since $x^2 \geqslant 0$ for all $x \in \mathbb{R}$. We have therefore shown that

$$a^2 + b^2 > 0,$$

which implies that $a^2 + b^2 \neq 0$, by the trichotomy property.

The argument for the case $b \neq 0$ is similar and the proof is now complete. □

2. Use induction to prove that $n > 0$ for all $n \in \mathbb{N}$.

Proof: Let $P(n)$ denote the statement "$n > 0$".

Observe that $1 > 0$ since $1 = 1^2 \geqslant 0$ and $1 \neq 0$. Thus, $P(1)$ is true.

Next, assume that $P(n)$ is true; that is, $n > 0$. We show that $P(n+1)$ is true.

Since $n > 0$ and $1 > 0$, it follows from the order Axiom O_2 that $n + 1 > 0$, which shows that $P(n+1)$ is true.

Hence, by the principle of mathematical induction, $n > 0$ for all $n \in \mathbb{N}$. □

3. Let r be a rational number satisfying $r > 0$. Prove that there exists a rational number, q, such that

$$0 < q < r.$$

Proof: Let $r \in \mathbb{Q}$ be positive. Then, $r = \dfrac{n}{m}$, where n and m are positive integers. Since, $m+1 > m$, it follows that

$$\frac{1}{m+1} < \frac{1}{m}.$$

Thus,

$$\frac{n}{m+1} < \frac{n}{m}.$$

since $n > 0$. We have therefore shown that

$$0 < \frac{n}{m+1} < r.$$

The proof follows by setting $q = \dfrac{n}{m+1}$. □

4. Let $a, b \in \mathbb{R}$. Suppose that $a < b + \varepsilon$ for every $\varepsilon > 0$. Prove that

$$a \leqslant b.$$

Proof: Argue by contradiction. Suppose that $a < b + \varepsilon$ for every $\varepsilon > 0$ and $a > b$. Then, $a - b > 0$. Set $\varepsilon = a - b$. By assumption

$$a < b + (a - b),$$

which implies that $0 < 0$. This is nonsense; therefore $a \leqslant b$. □

5. Let $x \in \mathbb{R}$. Prove that $0 \leqslant x < \varepsilon$ for every $\varepsilon > 0$ implies that $x = 0$.

Proof: Assume that $x \geqslant 0$ and $x < \varepsilon$ for all $\varepsilon > 0$. Then,

$$x < 0 + \varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Thus, by the result of the previous problem $x \leqslant 0$. Combining this with $x \geqslant 0$ yields that $x = 0$. □