Solutions to Assignment #5

1. Let a, b, c and d denote real numbers.

   Prove that if $a < b$ and $c < d$, then $a + c < b + d$.

   Proof: Assume that $a < b$ and $c < d$. Then, by the definition of order in $\mathbb{R}$,

   $$b - a > 0 \quad \text{and} \quad d - c > 0.$$

   It then follows from Axiom O_2 that

   $$b - a + d - c > 0,$$

   where we have used the associative property of addition. Thus, using associativity of addition again, commutativity of addition and the distributive property, we get that

   $$b + d - (a + c) > 0,$$

   which shows that

   $$a + c < b + d.$$

   □

2. For any real number a, show that $|-a| = |a|$.

   Proof: Suppose first that $a > 0$. Then, $-a < 0$, so that

   $$|-a| = -(-a) = a,$$

   by the definition of the absolute value function. Thus,

   $$|-a| = |a|,$$

   by the definition of absolute value again.

   Next, suppose that $a < 0$. Then, $-a > 0$, and so, by the definition of the absolute value,

   $$|-a| = -a = |a|,$$

   again by the definition of the absolute value.

   Finally, for $a = 0$, we also get $|-a| = |a|$ since $-0 = 0$ and $|0| = 0$.

   We have therefore proved that

   $$|-a| = |a| \quad \text{for all } a \in \mathbb{R}.$$

   □

3. Let a and b denote real numbers with $b \neq 0$. Show that

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

Proof: Let $a, b \in \mathbb{R}$ with $b \neq 0$. Then, b^{-1} exists. We first prove that

$$|b^{-1}| = \frac{1}{|b|}.$$

To see why this is the case, observe that

$$b^{-1}b = 1$$

so that

$$|b^{-1}b| = 1,$$

since $|1| = 1$, by the definition of absolute value, as $1 > 0$. Thus, by a result proved in class (see Problem 1(c) in Problem Set #2),

$$|b^{-1}||b| = 1,$$

from which we get that $|b|$ is invertible and

$$|b|^{-1} = |b^{-1}|,$$

which can be written as

$$|b^{-1}| = \frac{1}{|b|}. \tag{1}$$

Next, write

$$\frac{a}{b} = ab^{-1}.$$

and take the absolute value of both sides to get

$$\left|\frac{a}{b}\right| = |a||b^{-1}|, \tag{2}$$

where we have used again the result of Problem 1(c) in Problem Set #2. Consequently, using (1), we obtain from (2) that

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

which was to be shown. □

4. Prove that $|a+b+c| \leq |a|+|b|+|c|$ for all real numbers a, b and c.

   Proof: Apply the triangle inequality to

$$|a+b+c| = |(a+b)+c|$$

   to get

$$\begin{aligned} |a+b+c| &\leqslant |a+b|+|c| \\ &\leqslant |a|+|b|+|c|, \end{aligned}$$

   where we have used the triangle inequality a second time. □

5. Use induction on n to prove that

$$2^n > n \quad \text{for all } \ n \in \mathbb{N}.$$

   Proof: Let $P(n)$ denote the statement "$2^n > n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

   First note that $2^1 = 2 = 1+1 > 1$, since $1 > 0$. Consequently, $P(1)$ is true.

   Next, we prove the implication

$$P(n) \text{ is true } \ \Rightarrow P(n+1) \text{ is true.}$$

   Assume that $P(n)$ is true; that is, $2^n > n$. Consider

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n,$$

   and apply the assumption that $P(n)$ is true on the right hand side to get

$$2^{n+1} > n + n \geqslant n + 1,$$

   since $n \geqslant 1$, which shows that $P(n+1)$ is true.

   Hence, by induction on n, $2^n > n$ for all $n \in \mathbb{N}$. □