Solutions to Exam 1 (Part I)

1. Provide concise answers to the following questions:

   (a) Give the negation of the following statement:

   "For every $\varepsilon > 0$, there exists $n_o \in \mathbb{N}$ such that

   $$n \geqslant n_o \Rightarrow |x_n - x| < \varepsilon.''$$

   Answer: There exists $\varepsilon > 0$ such that, for all $n_o \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

   $$n \geqslant n_o \text{ and } |x_n - x| \geqslant \varepsilon.$$

   □

   (b) Let A denote a subset of the real numbers. Give the negation of the statement:

   "A is bounded above."

   Answer: For every real number, M, there exists an element, a, of A such that $a > M$. □

   (c) Let a and b denote real numbers. Give the contrapositive of the statement:

   "If $ab = 0$, then either $a = 0$ or $b = 0$."

   Answer: If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$. □

   (d) Let B denote a subset of the real numbers. Give the converse of the statement:

   "If B is non–empty and bounded below, then B has a greatest lower bound."

   Answer: If B has a greatest lower bound, then B is non–empty and bounded below. □

2. Let a denote a real number.

   (a) Use the field and order axioms of the real numbers to prove the statement:

   If $a > 1$, then $a^2 > 1$.

Proof: Assume that $a > 1$; then $a - 1 > 0$. Next, since $1 > 0$, it follows that $a > 0$; so that $a + 1 > 0$ by O_2. Consequently, by O_3,

$$(a + 1)(a - 1) > 0,$$

from which we get that $a^2 - 1 > 0$, or $a^2 > 1$, which was to be shown. □

(b) Use Mathematical Induction to prove the statement:

If $a > 1$, then $a^n > 1$ for all $n \in \mathbb{N}$.

Proof: Assume that $a > 1$. We show that $a^n > 1$ for all $n \in \mathbb{N}$ by induction on n.

The case $n = 1$ is true by the assumption that $a > 1$.

Next, we establish the implication: $a^n > 1 \Rightarrow a^{n+1} > 1$.

Assume that

$$a^n > 1. \tag{1}$$

Since we are assuming that $a > 1$, it is also the case that $a > 0$, since $1 > 0$. Multiply the inequality in (1) by a to obtain

$$a \cdot a^n > a \cdot 1,$$

from which we get that $a^{n+1} > a$, which implies that $a^{n+1} > 1$ since $a > 1$. The proof is now complete. □