Problem Set #1: The Set of Real Numbers

1. Let n be a natural number. Show that n is a multiple of 3 if and only if n^2 is a multiple of 3.

2. Show that $\sqrt{3}$ is irrational.

3. The following are consequences of the field axioms for the real numbers:

   (a) (*The cancelation laws*)

      (i) Let x, y and z be real numbers. If $x + z = y + z$, then $x = y$.

      (ii) Let x, y and z be real numbers with $z \neq 0$. If $xz = yz$, then $x = y$.

   (b) Show that 0 and 1 are unique.

   (c) Given $x \in \mathbb{R}$, $-x$ and x^{-1} are unique.

   (d) $x \cdot 0 = 0$ for all $x \in \mathbb{R}$.

   (e) Prove that $(-1)x = -x$, the additive inverse of x given by the field axiom (F_5), for every $x \in \mathbb{R}$.

   (f) $(-1) \cdot (-x) = x$ for all $x \in \mathbb{R}$.

4. Let a and b be real numbers. If $ab = 0$, then either $a = 0$ or $b = 0$.

5. Show that the set of rational numbers $\mathbb{Q}$ is a sub–field of the set of real numbers; that is, prove that for rational numbers q and r, $q + r \in \mathbb{Q}$ and $qr \in \mathbb{Q}$.

6. The following are consequences of the field and order axioms for the real numbers. Let x, y and z be real numbers.

   (a) If $x < y$ and $y < z$, then $x < z$.

   (b) If $x < y$, then $x + z < y + z$.

   (c) If $x < y$ and $z > 0$, then $xz < yz$.

   (d) If $x < 0$ and $y < 0$, then $xy > 0$.

   (e) If $x < y$ and $z < 0$, then $yz < xz$.

   (f) If $x < y$, then $-y < -x$.

   (g) $x^2 \geq 0$ for any real number x.

   (h) $1 > 0$

   (i) If $x > 0$, then $x^{-1} > 0$.

   (j) If $0 < x < y$, then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.