Problem Set #4: Completeness Axiom (Part II)

Read: Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Michael J. Schramm' book: "Introduction to Real Analysis."

Problems:

1. Let $x \in \mathbb{R}$ and define $A_x = \{m \in \mathbb{Z} \mid m \leq x\}$.

   (a) Prove that A_x is non–empty. Deduce that $\sup A_x$ exists and prove that there exist $n \in A_x$ such that $\sup A_x < n + 1$.

   (b) *(The Archimedean Property).* For any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.

2. Use the Archimedean Property established in part (b) of Problem 1 in this Problem Set to prove the following statements.

   (a) For every $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that $0 < \dfrac{1}{n} < \varepsilon$ for all $n \in \mathbb{N}$ such that $n \geq n_o$.

   (b) For every x and y in $\mathbb{R}$ such that $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $y < nx$.

3. Let x and y be real numbers satisfying $x < y$.

   (a) Prove that there exists $m \in \mathbb{N}$ such that $m(y - x) > 1$.

   (b) With m as given by part (a), prove that there exists $n \in \mathbb{Z}$ such that $n \leqslant mx < n + 1$.

   (c) With m and n given by parts (a) and (b), show that $mx < n + 1 < my$.

   (d) *(Density of $\mathbb{Q}$ in $\mathbb{R}$).* Prove that between any two real numbers there exits a rational number.

4. Let p be a positive real number. In this exercise we prove that there exists a real number x such that $x^2 = p$; that is, every positive real number has a square root.

   (a) Assume first that $p \geq 1$, and define $A = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^2 \leq p\}$. Prove that $\sup A$ exists.

   (b) Let $s = \sup A$ and show that $s^2 = p$; that is, s is a solution of $x^2 = p$ for $p \geq 1$.

   (c) Let $0 < p < 1$. Prove that $x^2 = p$ has a solution in $\mathbb{R}$.

   (d) Prove that for any positive number, p, there exists a unique positive solution to the equation $x^2 = p$.

5. Prove that $\mathbb{Q}$ is not a complete ordered field.