Assignment #6

Due on Friday, September 28, 2012

Read Section 4.1, *Continuous Functions,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

- *Definition of Continuous Function.* Let f be a real–valued function defined in a domain containing a. We say that f is continuous at a if

  $$\lim_{t\to a}|f(t)-f(a)| = 0.$$

  If f is continuous at every point in its domain, we say that f is continuous on that domain.

- *Absolute Value.* For any real number, x, the absolute value of x, denoted by $|x|$, is defined by $|x| = \begin{cases} x & \text{if } x \geqslant 0; \\ -x & \text{if } x < 0. \end{cases}$

Do the following problems

1. *The Triangle Inequality.*

   (a) Show that for any real numbers, x and y,

   $$|x+y| \leqslant |x| + |y|. \tag{1}$$

   Suggestion: Look at cases: (i) $x \leqslant y < 0$; (ii) $x < 0 \leqslant y$; (iii) $0 \leqslant x \leqslant y$.

   (b) Use (1) to show that, for any real numbers x and y,

   $$||x| - |y|| \leqslant |x-y|. \tag{2}$$

2. *The Absolute Value Function.* Define the real valued function f by $f(t) = |t|$ for all $t \in \mathbb{R}$; in other words, $f(t) = \begin{cases} t & \text{if } t \geqslant 0; \\ -t & \text{if } t < 0. \end{cases}$

   (a) Sketch the graph of $y = f(t)$.

   (b) Use the inequality in (2) to show that f is continuous on $\mathbb{R}$ and deduce that

   $$\lim_{t\to a}|t| = |a|,$$

   for all $a \in \mathbb{R}$.

3. Let $f(t) = |t^2 - 1|$ for t in $\mathbb{R}$.

   (a) Use the result of Problem 3 and facts about continuous functions presented in the class lecture notes to deduce that f is continuous on $\mathbb{R}$.

   (b) Sketch the graph of $y = f(t)$.

4. *The Square Root Function.* Set $f(t) = \sqrt{t}$ for $t \geqslant 0$.

   (a) Show that, if x and y are positive real numbers with $x < y$, then $\sqrt{x} < \sqrt{y}$.
   Suggestion: Use algebra to derive the equality

$$\sqrt{y} - \sqrt{x} = \frac{1}{\sqrt{y} + \sqrt{x}}(y - x). \tag{3}$$

   Observe that if the right–hand side of (3) is positive, then the left–hand side of (3) is also positive.

   (b) Deduce from part (a) that $f(t)$ increases as t increases over positive values and sketch the graph of $y = f(t)$.

5. *The Square Root Function (continued).* Set $f(t) = \sqrt{t}$ for $t \geqslant 0$.

   (a) Use algebra to show that, for $a > 0$,

$$|\sqrt{t} - \sqrt{a}| \leqslant \frac{1}{\sqrt{a}}|t - a|, \qquad \text{for } t \geqslant 0. \tag{4}$$

   Deduce (4) and the Squeeze Lemma that

$$\lim_{t \to a} \sqrt{t} = \sqrt{a}, \qquad \text{for } a > 0.$$

   Therefore, f is continuous at a for $a > 0$.

   (b) Let $\varepsilon > 0$ be an arbitrary positive number and put $\delta = \varepsilon^2$. Show that

$$0 \leqslant t < \delta \quad \text{implies that} \quad 0 \leqslant \sqrt{t} < \varepsilon. \tag{5}$$

   Explain your reasoning.

   (c) Explain why (5) says that $\sqrt{t}$ can be made arbitrarily small by making $t > 0$ sufficiently small. Deduce that

$$\lim_{t \to 0^+} \sqrt{t} = 0.$$

   Therefore, f is continuous at 0.