Solutions to Review Problems for Exam 1

1. Suppose that water leaks out a barrel at a constant rate. If the water level starts at 36 inches and drops to 35 inches in 30 seconds, how long will it take for the water to leak out of the barrel?

   Solution: Let $h(t)$ denote the height of the water level in the barrel at time t, where h is measured in inches and t in seconds. Since h is decreasing at a constant rate,
   $$h(t) = h(0) - rt, \qquad \text{for } t \geqslant 0, \tag{1}$$
   where $h(0) = 36$ is the initial water level, and $r = (36 - 35)/30$ inches per second, or
   $$r = \frac{1}{30} \text{ in/sec.}$$
   We then have from (1) that
   $$h(t) = 36 - \frac{1}{30}t, \qquad \text{for } t \geqslant 0, \tag{2}$$
   in inches and t is in seconds.

   We find the time, t, at which $h(t) = 0$. According to (2), this occurs when
   $$36 - \frac{1}{30}t = 0. \tag{3}$$
   Solving the equation in (3) for t yields
   $$t = 1,080 \text{ seconds,}$$
   or 18 minutes. Thus, it takes about eighteen minutes for the water to leak out of the barrel. □

2. Derive the identity $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, and use it to compute
   $$\lim_{n\to\infty} \frac{1}{n(n+1)}.$$

   Solution: Subtracting $\dfrac{1}{n+1}$ from $\dfrac{1}{n}$ yields
   $$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)},$$

so that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}. \tag{4}$$

We know that

$$\lim_{n\to\infty} \frac{1}{n} = 0. \tag{5}$$

Also, from $n+1 > n$ for all $n = 1, 2, 3, \ldots$, it follows that

$$0 < \frac{1}{n+1} < \frac{1}{n}, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{6}$$

It follows from (6), (5) and the Squeeze Lemma that

$$\lim_{n\to\infty} \frac{1}{n+1} = 0. \tag{7}$$

Hence, combining (4), (5) and (7), we get that

$$\lim_{n\to\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} \frac{1}{n+1} = 0 - 0 = 0.$$

$\square$

3. Use the fact that $\displaystyle\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$ to evaluate the following limits:

(a) $\displaystyle\lim_{t\to 0} \frac{\tan(t)}{t}$

***Solution**:* Write

$$\frac{\tan(t)}{t} = \frac{\sin t}{t} \cdot \frac{1}{\cos t}, \qquad \text{for } t \neq 0,$$

so that

$$\lim_{t\to 0} \frac{\tan(t)}{t} = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \frac{1}{\cos t} = 1 \cdot \frac{1}{1} = 1.$$

$\square$

(b) $\displaystyle\lim_{t\to 0} \frac{\sin^2(t)}{t}$

***Solution**:* Write

$$\frac{\sin^2(t)}{t} = \frac{\sin t}{t} \cdot \sin, \qquad \text{for } t \neq 0,$$

so that

$$\lim_{t\to 0} \frac{\sin^2(t)}{t} = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \sin t = 1 \cdot 0 = 0.$$

$\square$

4. Evaluate the following limits:

   (a) $\displaystyle\lim_{t\to 0}\frac{t^2-1}{t+1}$

   Solution: Compute

   $$\lim_{t\to 0}\frac{t^2-1}{t+1}=\frac{\lim\limits_{t\to 0}(t^2-1)}{\lim\limits_{t\to 0}(t+1)}=\frac{-1}{1}=-1.$$

   □

   (b) $\displaystyle\lim_{t\to\pi}\frac{\cos^2(t)}{t}$

   Solution: Compute

   $$\lim_{t\to\pi}\frac{\cos^2(t)}{t}=\frac{\lim\limits_{t\to\pi}[\cos t]^2}{\lim\limits_{t\to\pi} t}=\frac{(-1)^2}{\pi}=\frac{1}{\pi}.$$

   □

   (c) $\displaystyle\lim_{t\to -2}\frac{t+1}{t-1}$.

   Solution: Compute

   $$\lim_{t\to -2}\frac{t+1}{t-1}=\frac{\lim\limits_{t\to -2}(t+1)}{\lim\limits_{t\to -2}(t-1)}=\frac{-1}{-3}=\frac{1}{3}.$$

   □

5. Explain why each of the given functions, f, are continuous on $\mathbb{R}$.

   (a) $f(t)=\dfrac{\sin^3 t}{1+t^2}$ for all $t\in\mathbb{R}$.

   Solution: Note that $\sin^3$ is the composition of the polynomial function $p_1(u)=u^3$, for all $u\in\mathbb{R}$, the sine function, both of which are continuous functions on $\mathbb{R}$; hence $\sin^3$ is continuous on $\mathbb{R}$. Thus, f is a ratio of the function $\sin^3$ and the polynomial function $p_2(t)=t^2+1$ for all $t\in\mathbb{R}$. Since p_2 is continuous and $p_2(t)\neq 0$ for all $t\in\mathbb{R}$, it follows that

   $$f(t)=\frac{(p_1\circ\sin)(t)}{p_2(t)},\qquad\text{for all } t\in\mathbb{R},$$

   is continuous for all t in $\mathbb{R}$. □

(b) $f(t) = \begin{cases} t^2 \sin\left(\dfrac{1}{t}\right) & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$

***Solution**:* First note that, for $t \neq 0$, f is the product of the polynomial function $p(t) = t^2$, for $t \in \mathbb{R}$, and the composition of sin with the function, h, given by $h(t) = \dfrac{1}{t}$, for $t \neq 0$. All of these functions are continuous on their domain of definition; hence, f is continuous for $t \neq 0$.

For $t = 0$, we need to verify that

$$\lim_{t\to 0} f(t) = f(0) = 0;$$

in other words, we need to show that

$$\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = 0. \tag{8}$$

In order to establish (8), we use the inequality

$$|\sin\theta| \leqslant 1, \quad \text{for all } \theta \in \mathbb{R},$$

to obtain that

$$\left| t^2 \sin\left(\frac{1}{t}\right)\right| = t^2 \left|\sin\left(\frac{1}{t}\right)\right| \leqslant t^2, \quad \text{for } t \neq 0,$$

from which we get that

$$0 \leqslant \left| t^2 \sin\left(\frac{1}{t}\right)\right| \leqslant t^2, \quad \text{for } t \neq 0. \tag{9}$$

The limit calculation in (8) now follows from (9) by applying the Squeeze Lemma in conjunction with the limit facts

$$\lim_{t\to 0} 0 = 0 \quad \text{and} \quad \lim_{t\to 0} t^2 = 0.$$

$\square$

6. For the given function f, discuss the continuity or discontinuity of f at the given point a.

(a) $f(t) = |t-2|$ for all $t \in \mathbb{R}$ and $a = 2$.

***Solution**:* The function f is the composition of the absolute value function, $a(u) = |u|$ for all $u \in \mathbb{R}$, and the polynomial function $p(t) = t - 2$ for all $t \in \mathbb{R}$. Both of these functions are continuous on $\mathbb{R}$; hence, $f = a \circ p$ is continuous on $\mathbb{R}$. In particular, f is continuous at 2. $\square$

(b) $f(t) = \begin{cases} 1 & \text{if } t < 0; \\ -1 & \text{if } t \geqslant 0, \end{cases}$ and $a = 0$.

***Solution**:* Compute the one–sided limits

$$\lim_{t\to 0^-} f(t) = \lim_{t\to 0} 1 = 1 \quad \text{and} \quad \lim_{t\to 0^+} f(t) = \lim_{t\to 0} (-1) = -1,$$

so that

$$\lim_{t\to 0^-} f(t) \neq \lim_{t\to 0^+} f(t),$$

and therefore f has a jump discontinuity at $a = 0$. $\square$