Solutions to Review Problems for Exam 2

1. Let $f(t) = 0$ for $t < 0$, and $f(t) = 1 + t$ for $t \geqslant 0$, and let $A_f(0; x)$ denote the area under the graph of f from 0 to x.

   (a) Give a formula for computing $A_f(0; x)$ for all values of x.

   ***Solution**:* Figure 1 shows a sketch of the graph of f.

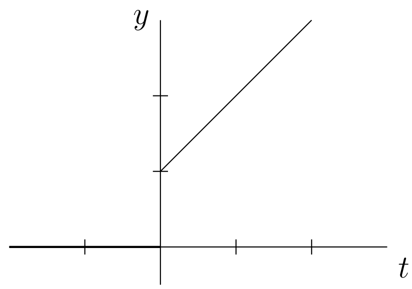


Figure 1: Sketch of graph of f

For $x < 0$,

$$A_f(0; x) = \int_0^x 0 \, dt = 0,$$

and, for $x \geqslant 0$,

$$\begin{aligned} A_f(0; x) &= \int_0^x (1 + t) \, dt \\ &= \left[t + \frac{1}{2} t^2 \right]_0^x \\ &= x + \frac{1}{2} x^2. \end{aligned}$$

We therefore have that

$$A_f(0; x) = \begin{cases} 0, & \text{if } x < 0; \\ x + \frac{1}{2} x^2, & \text{if } x \geqslant 0. \end{cases}$$

□

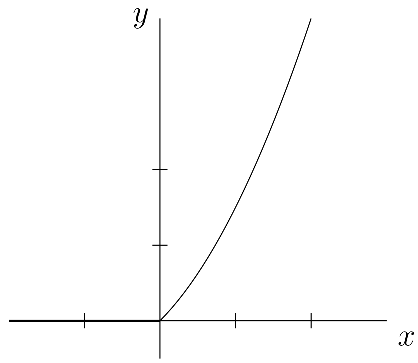


Figure 2: Sketch of graph of $y = A_f(0; x)$

(b) Sketch the graphs of $y = f(t)$ and $y = A_f(0; x)$.
***Solution**:* A sketch of the graph of $y = f(t)$ is shown in Figure 1.
A sketch of the graph of $y = A(0; x)$ is shown in Figure 2. □

2. Let $f(t) = \sqrt{t^4 + 1}$ for all $t \in \mathbb{R}$, and define $F(x) = \int_0^x f(t)\, dt$ for all $x \in \mathbb{R}$.

   (a) Explain why $F(x)$ increases as x increases.
   ***Solution**:* Note that $f(t) = \sqrt{t^4 + 1} \geqslant 1 > 0$, for all $t \in \mathbb{R}$; this, $f(t)$ is strictly positive for all t. Therefore, $F(x)$ increases with increasing x. □

   (b) Determine the values of x for which F is negative and those for which F is positive. Justify your answers.
   ***Solution**:* By the Sign Convention 1, since $f(t)$ is positive for all t, $F(x) > 0$ for $x > 0$, and $F(x) < 0$ for $x < 0$. □

3. Let $f(t) = |t| + 1$ for all $t \in \mathbb{R}$. Sketch the graph of $y = f(x)$ and evaluate the area under the graph of f from -3 to 3.

   ***Solution**:* Figure 3 shows a sketch of the graph of f.

   Note that the region, R, in question here is symmetric with respect to the y–axis. It then follows that

$$\text{area}(R) = 2\int_0^3 f(t)\, dt,$$

   so that

$$\text{area}(R) = 2\int_0^3 (t + 1)\, dt, \qquad (1)$$

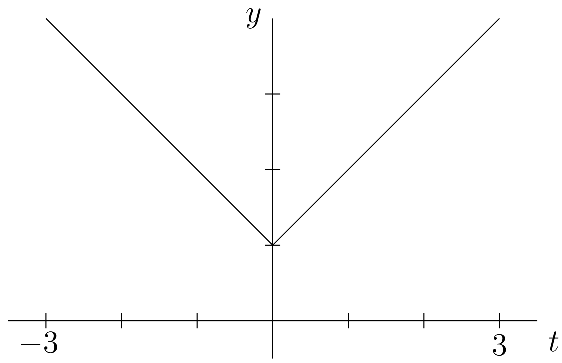


Figure 3: Sketch of graph of f

since $|t| = t$ for $t \geqslant 0$.

Evaluating the integral on the right–hand side of (1) we obtain

$$\begin{aligned} \text{area}(R) &= 2\left[\frac{1}{2}t^2 + t\right]_0^3 \\ &= 2\left(\frac{9}{2} + 3\right) \\ &= 15. \end{aligned}$$

□

4. Let $f(t) = \sqrt{1-(t-1)^2}$. Sketch the graph of $y = f(t)$ and evaluate the area under the graph of f from 0 to 1 that lies above the t–axis.

   ***Solution**:* A sketch of the graph of $y = f(t)$ is shown in Figure 4. The figure

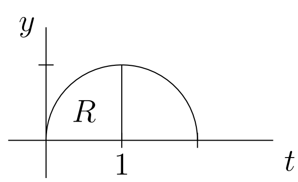


Figure 4: Sketch of graph of f

also shows the region, R, under consideration in this problem. Note that R is a quarter of a disc of radius 1, so that

$$\text{area}(R) = \frac{\pi}{4}.$$

□

5. Compute the area of the region in the ty–plane that lies below the line $y = t+2$ and above the graph of $y = t^2$.

Solution: Figure 5 shows a sketch of the region, R, under consideration in this problem. The region R lies below the graph of the line $y = t + 2$ and above

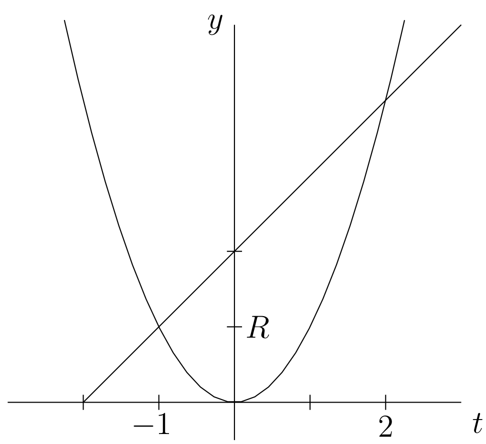


Figure 5: Sketch of region R

the graph of the parabola $y = t^2$ over an interval determined by the points of intersection of the line and the parabola. To find the points of intersection, solve the equation

$$t^2 = t + 2$$

to get $t = -1$ and $t = 2$. These points are labeled in Figure 5. We therefore get that

$$\text{area}(R) = \int_{-1}^{2} [t + 2 - t^2]\, dt. \tag{2}$$

Evaluating the integral on the right–hand side of (2) we get

$$\text{area}(R) = \left[\frac{1}{2}t^2 + 2t - \frac{1}{3}t^3\right]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right),$$

so that

$$\text{area}(R) = \frac{9}{2}.$$

□

6. Find the area of the region under the graph of $y = \dfrac{1}{\sqrt{t}}$ and above the t–axis from $t = 1$ to $t = 4$.

Solution: Figure 6 shows a sketch of the region R under consideration in this problem.

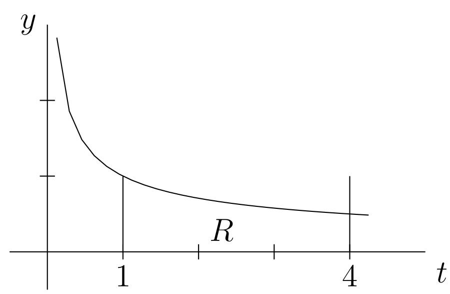


Figure 6: Sketch of region R

We have that

$$\text{area}(R) = \int_1^4 \frac{1}{\sqrt{t}}\, dt. \tag{3}$$

Writing $\dfrac{1}{\sqrt{t}} = t^{-1/2}$, we evaluate the integral on the right–hand side of (3) to get

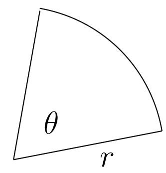
$$\text{area}(R) = \left[\frac{1}{-\frac{1}{2}+1} t^{-\frac{1}{2}+1}\right]_1^4,$$

or

$$\text{area}(R) = \left[2\sqrt{t}\right]_1^4 = 2\sqrt{4} - 2 = 2.$$

□

7. The area, A, of the circular sector shown in the figure



is given by the formula $A = \dfrac{1}{2}\theta r^2$, where θ is given in radians.

Use this formula to evaluate the integral $\displaystyle\int_0^1 \sqrt{4 - t^2}\, dt$.

***Solution**:* A sketch of the graph of $f(t) = \sqrt{4 - t^2}$, for $-2 \leqslant t \leqslant 2$, is shown in Figure 7. The definite integral $\displaystyle\int_0^1 \sqrt{4 - t^2}\, dt$ is the sum of the areas of

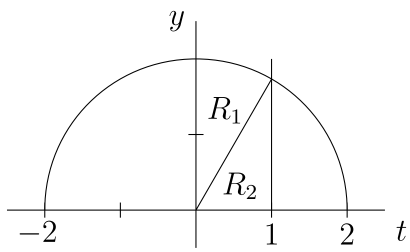


Figure 7: Sketch of graph of $y = \sqrt{4 - t^2}$, for $-2 \leqslant t \leqslant 2$

the circular sector R_1, pictured in the figure, and the triangle R_2, also pictured in the figure. The circular sector R_1 in Figure 7 is subtended by an angle, θ, satisfying

$$\sin\theta = \frac{1}{2},$$

so that

$$\theta = \frac{\pi}{6},$$

in radians. We then have that

$$\int_0^1 \sqrt{4 - t^2}\, dt = \text{area}(R_1) + \text{area}(R_2) = \frac{1}{2}\cdot\frac{\pi}{6}(2)^2 + \frac{1}{2}(1)\sqrt{3},$$

so that

$$\int_0^1 \sqrt{4 - t^2}\, dt = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

□

8. Let f be a function defined by $f(t) = \begin{cases} 0, & \text{if } t < -1 \\ \sqrt{1-t^2}, & \text{if } -1 \leqslant t < 0; \\ 1; & \text{if } t \geqslant 0. \end{cases}$

Evaluate the area function $F(x) = \displaystyle\int_{-1}^{x} f(t)\, dt$, for all $x \in \mathbb{R}$, and sketch the graph of $y = F(x)$.

Solution: A sketch of the graph of f is shown in Figure 8.

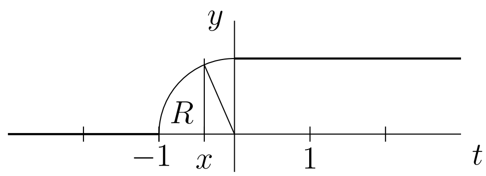


Figure 8: Sketch of graph of f

For $x < -1$, we compute

$$F(x) = \int_{-1}^{x} 0\, dt = 0. \tag{4}$$

For $-1 \leqslant x < 0$, refer to the region R in Figure 8. In this case we have that

$$\int_{-1}^{x} f(t)\, dt = \text{area}(R), \tag{5}$$

where area(R) is $\dfrac{\pi}{4}$ minus the area of a circular sector of radius 1 subtended by an angle, θ, given by $\sin\theta = -x$ plus the area of a triangle of base $-x$ and height $\sqrt{1-x^2}$. We then have that

$$\text{area}(R) = \frac{\pi}{4} - \left(\frac{1}{2}\arcsin(-x) + \frac{1}{2}(-x)\sqrt{1-x^2}\right),$$

so that

$$\text{area}(R) = \frac{\pi}{4} + \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}, \qquad \text{for } -1 \leqslant x < 0. \tag{6}$$

It follows from (5) and (6) that

$$F(x) = \frac{\pi}{4} + \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}, \qquad \text{for } -1 \leqslant x < 0. \tag{7}$$

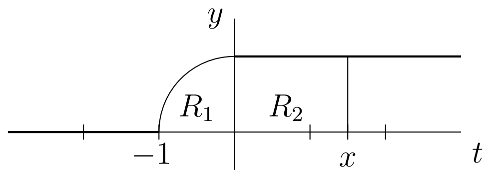


Figure 9: Sketch of graph of f

For the case $x > 0$, refer to Figure 9. In this case,

$$\int_{-1}^{x} f(t)\ dt = \text{area}(R_1) + \text{area}(R_2), \tag{8}$$

where R_1 is the the quarter of the circular disc depicted in Figure 9, and R_2 is the rectangle of width x and height 1, also depicted in the figure. It then follows from (8) that

$$\int_{-1}^{x} f(t)\ dt = \frac{\pi}{4} + x.$$

Consequently,

$$F(x) = \frac{\pi}{4} + x, \qquad \text{for } x \geqslant 0. \tag{9}$$

Combining (4), (7) and (9) we have that

$$F(x) = \begin{cases} 0, & \text{for } x < -1; \\ \frac{\pi}{4} + \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}, & \text{for } -1 \leqslant x < 0; \\ \frac{\pi}{4} + x, & \text{for } x \geqslant 0. \end{cases} \tag{10}$$

A sketch of the graph of $y = F(x)$, where F is as given in (10) is shown in Figure 10. □

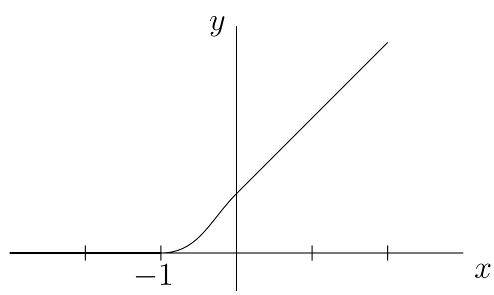


Figure 10: Sketch of graph of F