Exam 3

Friday, December 7, 2012

Name: _____

Show all significant work and justify all your answers. This is a closed book exam. Use your own paper and/or the paper provided by the instructor. You have 90 minutes to work on the following 5 problems. Relax.

1. Let $f\colon I \to \mathbf{R}$ denote a real valued function defined on some open interval, I, of real numbers. Let a denote any point in I.

   (a) State precisely what it means for f to be differentiable at a.

   (b) Use the definition of the derivative for an appropriately chosen function to show that the limit
   $$\lim_{h\to 0} \frac{\cos h - 1}{h}$$
   exists, and compute the limit. Explain your reasoning.

2. Let f denote a real valued function defined in some open interval I and let $a \in I$. Let F denote the area function
   $$F(x) = \int_a^x f(t)\, dt, \qquad \text{for } x \in I.$$

   (a) State precisely what the Second Fundamental Theorem of Calculus says about F.

   (b) Let $f(t) = \begin{cases} 1, & \text{if } t < 0; \\ 2t, & \text{if } t \geqslant 0, \end{cases}$ and put $F(x) = \displaystyle\int_0^x f(t)\, dt$, for all $x \in \mathbf{R}$. Compute $F(x)$, for $x \in \mathbf{R}$, and sketch the graph of $y = F(x)$.

   (c) Show that the function F defined in part (b) is not differentiable at 0. Explain why this does not contradict the Second Fundamental Theorem of Calculus.

3. Let $f(x) = \sqrt{x}$, for $x \geqslant 0$.

   (a) For $a > 0$, give the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, \sqrt{a})$.

   (b) Explain what happens to the slope of the tangent line to the graph of $y = \sqrt{x}$ at the point $(a, \sqrt{a})$, for $a > 0$, as $a \to 0^+$. What does this say about the the existence, or nonexistence, of a tangent line to the graph of $y = \sqrt{x}$ at $(0, 0)$

4. Let $F(x) = \displaystyle\int_0^x \frac{1}{\sqrt{1+t^4}}\, dt$, for $x \in \mathbf{R}$.

   (a) Explain why F is differentiable in $\mathbf{R}$ and compute F'.

   (b) Give the linear approximation to F at 0 and use it to estimate the integral

$$\int_0^{0.047} \frac{1}{\sqrt{1+t^4}}\, dt.$$

5. In each case, explain why the given function, f, is differentiable for all x in its domain, and compute f'.

   (a) $f(x) = \dfrac{x}{x^2+1}$ for all $x \in \mathbf{R}$.

   (b) $f(x) = \ln(\sqrt{1+x^2})$, for $x \in \mathbf{R}$.