Review Problems for Exam 3

1. Show that the limit $\lim\limits_{h\to 0} \dfrac{1}{h}\ln(1+h)$ exists and compute it.

2. Let $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$

   (a) Show that f is differentiable at 0 and compute $f'(0)$.

   (b) Explain why f is differentiable in $\mathbb{R}$ and compute f'.

3. Define $f(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$.

   (a) Explain why $f(x)$ exists for all $x \in \mathbb{R}$.

   (b) Explain why f is differentiable in $\mathbb{R}$ and compute f'.

4. Let $f(t) = |t|$ for all $t \in \mathbb{R}$ and put $F(x) = \displaystyle\int_0^x f(t)\, dt$, for all $x \in \mathbb{R}$.

   (a) Compute $F(x)$ for all $x \in \mathbb{R}$.

   (b) Explain why f is differentiable and compute F'.

5. Let f denote a continuous function defined in $\mathbb{R}$ and suppose that

$$\int_0^x f(t)\, dt = \sin(x^2), \qquad \text{for all } x \in \mathbb{R}.$$

   (a) Compute $f(x)$ for all $x \in \mathbb{R}$.

   (b) Explain why f is differentiable and compute f'.

6. Assume that g is continuous in $\mathbb{R}$ and define $G(x) = \displaystyle\int_1^x g(t)\, dt$, for all $x \in \mathbb{R}$. Evaluate each of the following in terms of G.

   (a) $\displaystyle\int_1^2 g(t)\, dt$.

   (b) $\displaystyle\int_{-2}^2 g(t)\, dt$.

7. Let $f(x) = \tan(x)$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

   (a) Give the equation of the tangent line to the graph of $y = f(x)$ at the point $(0, 0)$.

   (b) Give the linear approximation to f at $a = 0$ and use it to estimate $\tan(1°)$.

8. A rectangle has dimensions x and y. Assume that x and y are both differentiable functions of time, t. Let A denote the area of the rectangle.

   (a) Give a formula for computing the rate of change of A.

   (b) Given that, at time $t = 1$ the rectangle has dimensions $x = 4$ and $y = 7$, and that, at that instant, x is increasing at a rate of 0.3 units of length per second, and y is decreasing at a rate of 0.2 units of length per second, give the rate of change of area at $t = 1$.

9. Let f denote a continuous function define in $\mathbb{R}$ and put $g(x) = \displaystyle\int_2^{x^2} f(t)\, dt$, for all $x \in \mathbb{R}$. Explain why g is differentiable in $\mathbb{R}$ and compute g'.

10. Define $f(x) = \dfrac{\sqrt{4+x}}{2+\sqrt{x}}$, for $x \geqslant 0$.

    Explain why f is differentiable for $x > 0$ and compute f'.

11. Let $f(x) = \sin x$ for $x \in \mathbb{R}$. Compute the average value of f over the interval $[0, \pi]$.

12. A rod on length 2 meters in placed along the x–axis with its left–end at 0. Assume the material making up the rod has a linear density given by $\rho(x) = k\sqrt{1+x}$ (in grams per meter) for $0 \leqslant x \leqslant 2$, where k is a constant. Compute the mass of the rod.

13. Assume that f is a continuous function defined in $\mathbb{R}$ and that $2 + \displaystyle\int_a^x tf(t)\, dt = 2x^3$, for $x \in \mathbb{R}$. Find a and give a formula for computing $f(x)$, for all $x \in \mathbb{R}$.

14. Let $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$, for $x > 0$.

    (a) Explain why $\ln(x)$ is strictly increasing in x for all $x \in \mathbb{R}$.

    (b) Use the fact that $\ln(2^n) = n\ln(2)$ for all $n = 1, 2, 3, \ldots$ to explain why $\lim\limits_{x\to\infty} \ln(x) = +\infty$.

    (c) Use the fact that $\ln(2^{-n}) = -n\ln(2)$ for all $n = 1, 2, 3, \ldots$ to explain why $\lim\limits_{x\to 0^+} \ln(x) = -\infty$.

    (d) Explain why the natural logarithm function, ln, has an inverse function; that is, there exists $g\colon \mathbb{R} \to (0, \infty)$ such that

    $$g(\ln(x)) = x, \quad \text{for } x > 0 \quad \text{and} \quad \ln(g(x)) = x, \quad \text{for } x \in \mathbb{R}$$

    (e) Assuming that g is differentiable in $\mathbb{R}$, use the Chain Rule to give a formula for computing $g'(u)$ for all $u \in \mathbb{R}$.

15. Assume that oil is leaking from a tanker at a continuous rate, $R(t)$, in gallons per hour. Give a formula for computing the amount of oil that has leaked out of the tanker during the time interval $[0, t]$ for any $t \geqslant 0$.