Solutions to Review Problems for Exam 3

1. Show that the limit $\lim\limits_{h\to 0}\dfrac{1}{h}\ln(1+h)$ exists and compute it.

   ***Solution**:* Let $f(x)=\ln(1+x)$, for $x>-1$, and note that $\dfrac{1}{h}\ln(1+h)$ is the difference quotient of f at 0; that is

   $$\frac{1}{h}\ln(1+h)=\frac{f(0+h)-f(0)}{h},\qquad \text{for } h\neq 0. \tag{1}$$

   Note that f is a composition the natural logarithm function, ln, and the polynomial function $p(x)=1+x$; that is, $f(x)=\ln(p(x)$, for $x>-1$. Note also that $p(x)>0$ for $x>-1$. Thus, since both ln and p are differentiable, it follows that f is differentiable and, by the Chain Rule,

   $$f'(x)=\ln'(p(x))\cdot p'(x),\qquad \text{for } x>-1,$$

   from which we get that

   $$f'(x)=\frac{1}{x+1},\qquad \text{for } x>-1. \tag{2}$$

   Thus, f is differentiable at 0, so that the limit as $h\to 0$ of the difference quotient in (1) exists and

   $$\lim_{h\to 0}\frac{1}{h}\ln(1+h)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=f'(0)=1,$$

   where we have used (2). □

2. Let $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$

   (a) Show that f is differentiable at 0 and compute $f'(0)$.
   ***Solution**:* First, compute the difference quotient of f at 0 to get

   $$\frac{f(0+h)-f(0)}{h}=h\sin\left(\frac{1}{h}\right),\qquad \text{for } h\neq 0. \tag{3}$$

   Next, take absolute values of both sides of (3) and use the fact that $|\sin(t)|\leqslant 1$ for all $t\in\mathbb{R}$, to get that

   $$0\leqslant\left|\frac{f(0+h)-f(0)}{h}\right|\leqslant |h|,\qquad \text{for } h\neq 0. \tag{4}$$

It follows from (4) and the Squeeze Lemma that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=0,$$

so that f is differentiable at 0 and $f'(0)=0$. □

(b) Explain why f is differentiable in $\mathbb{R}$ and compute f'.

***Solution**:* We have already seen in part (a) that f is differentiable at 0 and that $f'(0)=0$. It remains to consider the case $x\neq 0$.

If $x\neq 0$,

$$f(x)=x^2\sin\left(\frac{1}{x}\right), \tag{5}$$

which displays f as a product of differentiable functions (the second factor in (5) being a composition of differentiable functions for $x\neq 0$).

Applying the Product Rule and the Chain Rule, we obtain from (5) that

$$f'(x)=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right),\qquad \text{for } x\neq 0.$$

Thus,

$$f'(x)=\begin{cases}2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right), & \text{for } x\neq 0;\\ 0, & \text{for } x=0.\end{cases}$$

□

3. Define $f(x)=\displaystyle\int_0^x\frac{\sin t}{t}\,dt$.

(a) Explain why $f(x)$ exists for all $x\in\mathbb{R}$.

***Solution**:* Note that integrand in the definition of f has has removable singularity at 0 because

$$\lim_{t\to 0}\frac{\sin t}{t}=1.$$

Thus, defining

$$g(t)=\begin{cases}\dfrac{\sin t}{t}, & \text{if } t\neq 0;\\ 1, & \text{if } t=0,\end{cases} \tag{6}$$

we see that

$$f(x)=\int_0^x g(t)\,dt,\qquad \text{for } x\in\mathbb{R}, \tag{7}$$

where g is continuous in $\mathbb{R}$. It follows then, from the Existence of Area Function Theorem, that $f(x)$ exists for all $x \in \mathbb{R}$.

□

(b) Explain why f is differentiable in $\mathbb{R}$ and compute f'.

***Solution**:* Since the function g defined in (6) is continuous, it follows from (7) and the Second Fundamental Theorem of Calculus that f is differentiable and

$$f'(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

□

4. Let $f(t) = |t|$ for all $t \in \mathbb{R}$ and put $F(x) = \displaystyle\int_0^x f(t)\ dt$, for all $x \in \mathbb{R}$.

(a) Compute $F(x)$ for all $x \in \mathbb{R}$.

***Solution**:* Compute

$$F(x) = \int_0^x |t|\ dt, \qquad \text{for } x \in \mathbb{R}.$$

If $x < 0$, we have that

$$F(x) = \int_0^x -t\ dt = -\frac{x^2}{2}.$$

If $x \geqslant 0$, we have that

$$F(x) = \int_0^x t\ dt = \frac{x^2}{2}.$$

We therefore have that

$$F(x) = \begin{cases} -\dfrac{x^2}{2}, & \text{if } x < 0; \\ \\ \dfrac{x^2}{2}, & \text{if } x \geqslant 0. \end{cases}$$

□

(b) Explain why f is differentiable and compute F'.

Solution: Observe that the function f is continuous in $\mathbb{R}$. Thus, by the Second Fundamental Theorem of Calculus, F is differentiable and

$$F'(x) = f(x) = |x|, \quad \text{for all } x \in \mathbb{R}.$$

□

5. Let f denote a continuous function defined in $\mathbb{R}$ and suppose that

$$\int_0^x f(t)\, dt = \sin(x^2), \quad \text{for all } x \in \mathbb{R}. \tag{8}$$

(a) Compute $f(x)$ for all $x \in \mathbb{R}$.

Solution: Since f is continuous in $\mathbb{R}$, we can apply the Second Fundamental Theorem of Calculus to differentiate the left–hand side of the equation in (8) to get

$$f(x) = \frac{d}{dx}[\sin(x^2)] = 2x\cos(x^2), \quad \text{for } x \in \mathbb{R}, \tag{9}$$

where we have applied the Chain Rule. □

(b) Explain why f is differentiable and compute f'.

Solution: In view of (9), we see that f is a product of differentiable functions. Hence, f is differentiable and, using the Product Rule and the Chain Rule,

$$f'(x) = 2\cos(x^2) - 4x^2\sin(x^2), \quad \text{for } x \in \mathbb{R}.$$

□

6. Assume that g is continuous in $\mathbb{R}$ and define $G(x) = \displaystyle\int_1^x g(t)\, dt$, for all $x \in \mathbb{R}$. Evaluate each of the following in terms of G.

(a) $\displaystyle\int_1^2 g(t)\, dt$.

Solution: By the definition of G, $\displaystyle\int_1^2 g(t)\, dt = G(2)$. □

(b) $\displaystyle\int_{-2}^2 g(t)\, dt$.

Solution: By the Second and Third Fundamental Theorems of Calculus, $\displaystyle\int_1^2 g(t)\, dt = G(2) - G(-2)$. □

7. Let $f(x) = \tan(x)$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

   (a) Give the equation of the tangent line to the graph of $y = f(x)$ at the point $(0, 0)$.

   ***Solution**:* First, apply the Quotient Rule to $f(x) = \dfrac{\sin(x)}{\cos(x)}$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, to get that

$$f'(x) = \frac{1}{\cos^2 x} = \sec^2(x), \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

   The equation of the tangent line to the graph of $y = \tan(x)$ at $(0, 0)$ is given by

$$y = f(0) + f'(0)(x - 0),$$

   or

$$y = x.$$

   □

   (b) Give the linear approximation to f at $a = 0$ and use it to estimate $\tan(1^\circ)$.
   The linear approximation to $\tan(x)$ at $a = 0$ is

$$L(x) = f(0) + f'(0)(x - 0) = x, \quad \text{for all } x \in \mathbb{R}.$$

   Thus,

$$\tan(x) \approx x, \quad \text{for } x \text{ close to } 0.$$

   In particular,

$$\tan(1^\circ) = \tan\left(\frac{\pi}{180}\right) \approx \frac{\pi}{180} \doteq 0.0175.$$

8. A rectangle has dimensions x and y. Assume that x and y are both differentiable functions of time, t. Let A denote the area of the rectangle.

   (a) Give a formula for computing the rate of change of A.

   ***Solution**:* The area of the rectangle, as a function of t, is given by

$$A(t) = x(t)y(t), \quad \text{for } t \geqslant 0.$$

   Thus, by the Product Rule, the rate of change of A is

$$\frac{dA}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}y. \tag{10}$$

   □

(b) Given that, at time $t = 1$ the rectangle has dimensions $x = 4$ and $y = 7$, and that, at that instant, x is increasing at a rate of 0.3 units of length per second, and y is decreasing at a rate of 0.2 units of length per second, give the rate of change of area at $t = 1$.

Solution: Apply the result in (10) to $x = 4$, $y = 7$, $\dfrac{dx}{dt} = 0.3$ and $\dfrac{dx}{dt} = -0.2$, we get that, at $t = 1$,

$$\frac{dA}{dt} = 4(-0.2) + 7(0.3) = 2.1 - 0.8 = 1.3.$$

Thus, at $t = 1$, the area is increasing at a rate of 1.3 units of area per second. □

9. Let f denote a continuous function define in $\mathbb{R}$ and put $g(x) = \displaystyle\int_2^{x^2} f(t)\,dt$, for all $x \in \mathbb{R}$. Explain why g is differentiable in $\mathbb{R}$ and compute g'.

Solution: Put $F(u) = \displaystyle\int_2^{u} f(t)\,dt$, for $u \in \mathbb{R}$. Then, since f is continuous, it follows from the Second Fundamental Theorem of Calculus that the F is differentiable and

$$F'(u) = f(u), \qquad \text{for all } u \in \mathbb{R}. \tag{11}$$

Observe that $g(x) = F(x^2)$, for all $x \in \mathbb{R}$, so that g is the composition of F and the polynomial function $p(x) = x^2$, for $x \in \mathbb{R}$, both of which are differentiable. It follows that g is differentiable and, by the Chain Rule,

$$g'(x) = F'(x^2)p'(x) = f(x^2)(2x) = 2xf(x^2), \qquad \text{for } x \in \mathbb{R}.$$

□

10. Define $f(x) = \dfrac{\sqrt{4+x}}{2+\sqrt{x}}$, for $x \geqslant 0$.

Explain why f is differentiable for $x > 0$ and compute f'.

Solution: f is the ratio of two differentiable functions for $x > 0$, where the denominator is not zero for $x > 0$. Hence, f is differentiable and, by the

Quotient Rule,

$$
\begin{aligned}
f'(x) &= \frac{(2+\sqrt{x})\dfrac{d}{dx}[\sqrt{4+x}] - \sqrt{4+x}\dfrac{d}{dx}[2+\sqrt{x}]}{(2+\sqrt{x})^2} \\
&= \frac{(2+\sqrt{x})\dfrac{1}{2\sqrt{4+x}} - \sqrt{4+x}\dfrac{1}{2\sqrt{x}}}{(2+\sqrt{x})^2} \\
&= \frac{(2+\sqrt{x})\sqrt{x} - (4+x)}{2\sqrt{x}\sqrt{4+x}(2+\sqrt{x})^2} \\
&= \frac{2\sqrt{x} + x - 4 - x}{2\sqrt{x}\sqrt{4+x}(2+\sqrt{x})^2},
\end{aligned}
$$

so that

$$
f'(x) = \frac{\sqrt{x}-2}{\sqrt{x}\sqrt{4+x}(2+\sqrt{x})^2}, \qquad \text{for } x > 0.
$$

□

11. Let $f(x) = \sin x$ for $x \in \mathbb{R}$. Compute the average value of f over the interval $[0, \pi]$.

    ***Solution**:* The average value of f over $[0, \pi]$ is given by

$$
\overline{f} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx = \frac{2}{\pi}.
$$

□

12. A rod on length 2 meters in placed along the x–axis with its left–end at 0. Assume the material making up the rod has a linear density given by $\rho(x) = k\sqrt{1+x}$ (in grams per meter) for $0 \leqslant x \leqslant 2$, where k is a constant. Compute the mass of the rod.

***Solution**:* The mass of the rod is given by

$$
\begin{aligned}
M &= \int_0^2 \rho(x)\ dx \\
&= \int_0^2 k\sqrt{1+x}\ dx \\
&= k\left[\frac{2}{3}(1+x)^{3/2}\right]_0^2 \\
&= \frac{2k}{3}\left[3^{3/2}-1\right].
\end{aligned}
$$

□

13. Assume that f is a continuous function defined in $\mathbb{R}$ and that $2+\int_a^x tf(t)\ dt = 2x^3$, for $x\in\mathbb{R}$. Find a and give a formula for computing $f(x)$, for all $x\in\mathbb{R}$.

    ***Solution**:* Differentiate with respect to x on both sides of the equation

    $$2+\int_a^x tf(t)\ dt = 2x^3, \quad \text{for } x\in\mathbb{R}, \tag{12}$$

    to get

    $$\frac{d}{dx}\int_a^x tf(t)\ dt = \frac{d}{dx}[2x^3], \quad \text{for } x\in\mathbb{R}, \tag{13}$$

    Since f is assumed to be continuous, we can apply the Second Fundamental Theorem of Calculus on the left–hand side of (13) to get that

    $$xf(x) = 6x^2, \quad \text{for } x\in\mathbb{R}. \tag{14}$$

    Solving for $f(x)$ in (14) yields

    $$f(x) = 6x, \quad \text{for } x\in\mathbb{R}.$$

    To find a, substitute a for x in (12) to get

    $$2+0 = 2a^3,$$

    which yields $a^3 = 1$, from which we get $a = 1$. □

14. Let $\ln(x) = \int_1^x \frac{1}{t}\ dt$, for $x>0$.

(a) Explain why $\ln(x)$ is strictly increasing in x for all $x > 0$.

***Solution**:* Observe that $\dfrac{1}{t} > 0$ for $t > 0$, so $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$ is strictly increasing with $x > 0$. □

(b) Use the fact that $\ln(2^n) = n\ln(2)$ for all $n = 1, 2, 3, \ldots$ to explain why $\lim\limits_{x\to\infty} \ln(x) = +\infty$.

***Solution**:* To see that $\ln(x)$ tends to infinity as $x \to \infty$, observe that if $x > 2^n$, then since ln is strictly increasing,

$$\ln(x) > \ln(2^n),$$

or

$$\ln(x) > n\ln(2). \tag{15}$$

Thus, according to (14), since $\ln(2) > \ln(1) = 0$, we can make the right–hand side of (15) arbitrarily large by taking n large. □

(c) Use the fact that $\ln(2^{-n}) = -n\ln(2)$ for all $n = 1, 2, 3, \ldots$ to explain why $\lim\limits_{x\to 0^+} \ln(x) = -\infty$.

***Solution**:* To see that $\ln(x)$ tends to $-\infty$ as $x \to 0^+$, observe that if $0 < x < 2^{-n}$, then since ln is strictly increasing,

$$\ln(x) < \ln(2^{-n}),$$

or

$$\ln(x) < -n\ln(2). \tag{16}$$

Thus, according to (16), $\ln(x)$ can be made to go to $-\infty$ by making n larger and larger. □

(d) Explain why the natural logarithm function, ln, has an inverse function; that is, there exists $g\colon \mathbb{R} \to (0, \infty)$ such that

$$g(\ln(x)) = x, \quad \text{for } x > 0 \quad \text{and} \quad \ln(g(x)) = x, \quad \text{for } x \in \mathbb{R} \tag{17}$$

***Solution**:* It follows from parts (a), (b) and (c) that $\ln\colon (0, \infty) \to (-\infty, +\infty)$ is one–to–one and onto. Hence, ln has inverse function $g\colon (-\infty, +\infty) \to (0, +\infty)$ satisfying (17). □

(e) Assuming that g is differentiable in $\mathbb{R}$, use the Chain Rule to give a formula for computing $g'(u)$ for all $u \in \mathbb{R}$.

***Solution**:* Set $u = \ln(x)$ for $x > 0$ and use the first equation in (17) to get

$$g(u) = x, \qquad \text{for } u = \ln(x), \quad x > 0. \tag{18}$$

Applying the Chain Rule to the first equation in (18) we get

$$g'(u)\frac{du}{dx} = 1, \tag{19}$$

where

$$\frac{du}{dx} = \frac{1}{x} = \frac{1}{g(u)}, \tag{20}$$

where we have used the first equation in (18) again.

Combining (19) and (20) we get

$$g'(u)\frac{1}{g(u)} = 1,$$

from which we get that

$$g'(u) = g(u), \qquad \text{for all } u \in \mathbb{R}.$$

□

15. Assume that oil is leaking from a tanker at a continuous rate, $R(t)$, in gallons per hour. Give a formula for computing the amount of oil that has leaked out of the tanker during the time interval $[0, t]$ for any $t \geqslant 0$.

    ***Solution**:* Let $Q(t)$ denote the amount of oil that has leaked out of the tanker since time $t = 0$. Then,

    $$Q(t) = \int_0^t R(\tau)\ d\tau, \qquad \text{for } t \geqslant 0.$$

    □