## Review Problems for Final Exam

1. Water flows from the bottom of a tank at a rate of $R(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$ in minutes. Find the amount of water that flows from the tank during the last 20 minutes.

2. The acceleration to earth's gravitation, $g$, at a point a distance $r$ from the center of the earth is a function of the distance, $r$, given by

$$g(r) = \begin{cases} \dfrac{kr}{R^3}, & \text{for } r < R; \\ \\ \dfrac{k}{r^2}, & \text{for } r \geqslant R, \end{cases}$$

   where $R$ is the radius of the earth and $k$ is some positive constant.

   (a) Explain why $g$ is continuous on $[0, \infty)$.

   (b) Discuss the differentiability properties of $g$. Is $g$ differentiable in $[0, \infty)$? If not, where does it fail to be differentiable?

3. Let $f(x) = \begin{cases} \dfrac{\sin(x)}{|x|}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

   Determine whether or not $f$ is continuous in $\mathbb{R}$. If not, determine the point of discontinuity and their type of discontinuity.

4. Let $f(x) = \begin{cases} mx + 1, & \text{if } x < 1; \\ x^2, & \text{if } x \geqslant 1. \end{cases}$

   Determine the value of $m$ that will make the function $f$ continuous in $\mathbb{R}$. Explain your reasoning.

5. Let $f(x) = \ln x$ for $x > 0$.

   (a) Give the linear approximation to $f$ at $a = 1$.

   (b) Use the linear approximation to ln at 1 in order to estimate $\ln(0.95)$.

6. Let $f(x) = \begin{cases} 0, & \text{if } x < 0; \\ x^2, & \text{if } x \geqslant 0. \end{cases}$

   (a) Show that $f$ is differentiable at 0 and compute $f'(0)$.

   (b) Explain why $f$ is differentiable in $\mathbb{R}$ and compute $f'$.

   (c) Is $f'$ differentiable at 0? Justify your answer.

7. Give the equation of the tangent line to the graph of $y = \dfrac{2x}{1+x^2}$ at the point $(-1,-1)$. Give the $x$–intercept and $y$–intercept of the tangent line.

8. Give an example of a function, $f$, that is continuous on the interval $[-2,1]$, but is not differentiable at $-1$. Justify your answer.

9. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a real–valued function that satisfies $f(x+y)=f(x)+f(y)$ for all $x$ and $y$ in $\mathbb{R}$.

   (a) Show that $f(0)=0$.

   (b) Assume, in addition, that $\displaystyle\lim_{h\to 0}\frac{f(h)}{h}=47$. Show that $f$ is differentiable in $\mathbb{R}$ and compute $f'$. Give a formula for computing $f(x)$ for all $x\in\mathbb{R}$.

10. Let $f(x)=2\sqrt{x+1}$ for $x\geqslant -1$.

    (a) Explain why $f$ is differentiable for $x>-1$ and compute $f'(x)$ for $x>-1$.

    (b) Give a formula for evaluating the indefinite integral $\displaystyle\int\frac{1}{\sqrt{x+1}}\,dx$.

11. Compute the area of the region bounded by the graphs of $y=\sin x$ and $y=\cos x$ between the lines $x=1$ and $x=3$. Note: Give the exact value of the area.

12. Let $f(x)=\ln(\sqrt{1+x})-\ln(\sqrt{1-x})$ for $-1<x<1$.

    (a) Explain why $f$ is differentiable in the open interval $(-1,1)$, and compute $f'(x)$ for $-1<x<1$.

    (b) Evaluate the definite integral $\displaystyle\int_{-1/2}^{1/2}\frac{1}{1-x^2}\,dx$. Give the exact value.

13. Let $f(x)=\cos(2x)+2\sin^2(x))$ for $x\in\mathbb{R}$.

    (a) Explain why $f$ is differentiable in $\mathbb{R}$ and compute $f'$.

    (b) Use the trigonometric identity $\sin(2\theta)=2\sin\theta\cos\theta$ to simplify the expression for $f'$ in part (a). What do you conclude about $f$?

14. Let $a>0$ and define $F(x)=\displaystyle\int_1^{ax}\frac{1}{t}\,dt$, for $x>0$.

    (a) Explain why $F$ is differentiable for $x>0$ and compute $F'(x)$ for $x>0$.

    (b) Put $g(x)=F(x)-\ln x$ for all $x>0$. Explain why $g$ is differentiable in $(0,+\infty$ and compute $g'(x)$ for $x>0$. What do you conclude about $g$? What does this say about $F(x)$, for $x>0$.