Calculus I

Preliminary Lecture Notes

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Chapter 1

Preface

This set of notes has been developed in conjunction with the teaching of Math 30 (Calculus I) at Pomona College during the fall semester of 2012. The course is an introduction to integral and differential Calculus. No previous knowledge of Calculus will be assumed. However, a good working knowledge of algebra and elementary functions are essential for a successful enjoyment of the course.

There are three major goals in the course: (1) the acquisition of a thorough understanding of the concepts and ideas of integral and differential Calculus; (2) the development of an appreciation for the power of Calculus in solving real world problems, and the mastery of several of the tools from Calculus that are very useful in applications; (3) the improvement of formal reasoning and problem solving skills.

Various applications will be used to motivate the concepts and as a source of interesting problems. We will begin with a discussion of a problems which is very common in applications: Suppose that we have information about the rate of change of a function for all time. Can we use this information to reconstruct the function? In the next section we illustrate the ideas that go into answering this question for the case in which we know the speed of a vehicle as a function of time, and we want to find out the distance traveled by the vehicle in any time interval.

Chapter 2

Introductory Example: Recovering a Function from its Rate of Change

In this introductory example we illustrate the use of two important ideas in Calculus: limits or approximations, and continuity. In later sections in these notes, we expand on these ideas in more detail and we will see how they can be used in several applications.

2.1 Recovering Distance from Speed

Imagine that the odometer in your car is broken. However, the computer in your car is able to keep track of the speed of your car at any time, t, where t is measured in hours. We therefore obtain the speed, v, of the vehicle as a function of t, $v(t)$, where v is measured in miles per hour. The picture in Figure 2.1.1 shows a possible graph of the speed, v, as a function of time, t. The figure

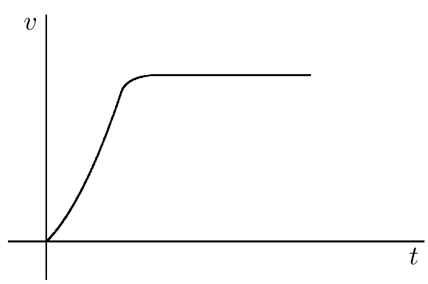

Figure 2.1.1: Graph of speed versus time

shows a situation in which the vehicle accelerates from rest to a certain speed, and travels at that speed for certain interval of time.

We would like to determine the distance traveled by the vehicle from the time it is at rest and starts moving, call that time t_o, to any given time t. If we are able to do this, we obtain a function of time, $s(t)$, which gives the distance traveled by the vehicle from time t_o to time t.

$$s(t) = \text{ distance traveled over the time interval } [t_o, t]. \tag{2.1}$$

Having introduced the function s defined in (2.1), we can reformulate the problem that we are studying in this section as follows:

Problem 2.1.1. *Given the speed, $v(t)$, of a vehicle at any time $t \geqslant t_o$, compute the distance traveled by the vehicle over the interval of time $[t_o, t]$.*

Note that since the speed of the vehicle is the rate of change of distance from a given point, Problem 2.1.1 is an instance of a problem in which we are interested in computing a given function based on information provided by its rate of change.

Example 2.1.2. The simplest example of Problem 2.1.1 is provided by the situation in which the vehicle's speed is constant; say,

$$v(t) = c, \qquad \text{for all } t \geqslant t_o, \tag{2.2}$$

where the positive constant c has units of miles per hour. If t is measured in hours, then the distance traveled by the vehicle from time t_o to time t is given by

$$s(t) = c(t - t_o), \qquad \text{for all } t \geqslant t_o. \tag{2.3}$$

Figure 2.1.2 shows the graph of the constant speed function in (2.2) over the

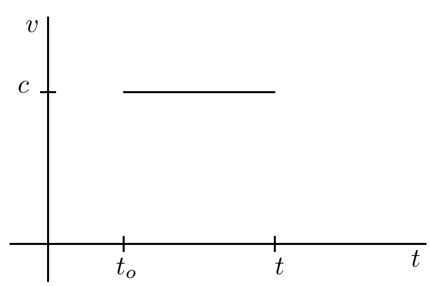

Figure 2.1.2: Graph of speed in (2.2)

interval $[t_o, t]$. The graph of the distance function in (2.3) in shown in Figure 2.1.3.

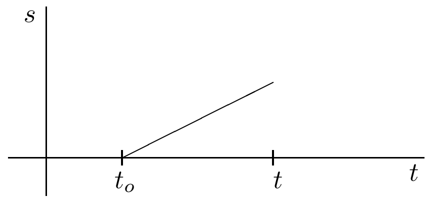

Figure 2.1.3: Graph of $s(t)$ in (2.3)

The result of Example 2.1.2 illustrates what happens in the general case of a function whose rate of change with respect to time is constant. Suppose that the rate of change of a function, f, is a constant, c, then

$$f(t) = f(t_o) + c(t - t_o), \quad \text{for all } t. \tag{2.4}$$

In order to understand what the expression in (2.4) is saying, rewrite it as

$$\frac{f(t) - f(t_o)}{t - t_o} = c, \quad \text{for all } t \neq t_o. \tag{2.5}$$

The quantity on the left of (2.5) measures that change in the amount of f over the interval $t - t_o$ per unit of time. The assumption that f has a constant rate of change is the statement that the expressions on left of (2.5) are the same for all $t \neq t_o$.

Observe that the expression for $s(t)$ in (2.3) is a special case of (2.4) in which $f(t) = s(t)$ and $s(t_o) = 0$, since $s(t)$ measures the distance traveled over the time interval $[t_o, t]$.

The case in which the rate of a change of a function varies with time is more complicated. In the remainder of this section we will introduce the ideas that go into solving this problem. These ideas at are at the core of the development of the integral and differential Calculus that we will be discussion in this course.

Example 2.1.3. In this example we consider the case in which the speed of the vehicle varies with time in a manner that is proportional to the time elapsed. Take $t_o = 0$ and assume that

$$v(t) = at \qquad \text{for all } t, \tag{2.6}$$

where a is a positive constant of proportionality that has units of miles/hr^2. Thus, (2.6) describes a situation in which the vehicle accelerates from rest to the speed of at mph after t hours. Figure 2.1.4 shows the graph of $v(t)$ in (2.6).

We would like to compute the distance that the vehicle travels over the interval $[0, t]$ if the speed is given by the function in (2.6). We present here a few ideas that can be used to solve this problem.

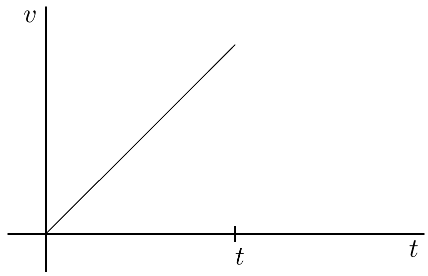

Figure 2.1.4: Graph of speed in (2.6)

Note that v is not constant over the interval $[0, t]$, so the procedure used to solve the problem in Example 2.1.2 cannot be used here. We can however, subdivide the time interval $[0, t]$ into n small subintervals

$$[0, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \quad \text{where } t_n = t. \tag{2.7}$$

We assume that n is a very large positive integer. We may also assume that all the subintervals in (2.7) have the same length, so that

$$t_k - t_{k-1} = \frac{t}{n}, \quad \text{for all } k = 1, 2, 3, \ldots, n. \tag{2.8}$$

Thus, as n increases the length of the intervals in (2.7) decreases. In fact, we see from (2.8) that the lengths of the subintervals tends to 0. We express this in symbols as

$$\lim_{n\to\infty} (t_k - t_{k-1}) = \lim_{n\to\infty} \frac{t}{n} = 0. \tag{2.9}$$

The idea expressed in (2.9) is that of a limit. Later in these notes, we will spend some time elaborating on the concept of limit and using this concept to make calculations.

The next idea that we will use when solving the problem in Example 2.1.3 is that, since the length of the interval $[t_{k-1}, t_k]$ is very small when n is large, then we may assume that $v(t)$ is approximately constant on that interval. In symbols we may write

$$v(t) \approx v(\tau_k), \qquad \text{for } t_{k-1} \leqslant t \leqslant t_k, \tag{2.10}$$

where τ_k is any time in the interval $[t_{k-1}, t_k]$. The idea encapsulated in the expression in (2.10) is that of continuity of the function v. Thus, in the solution to the general problem of this section, we will assume that the function v is continuous. The idea of continuity will be another major theme in this course.

Thus, if n is very large, we can use (2.10) and the result of Example 2.1.2 to approximate the distance traveled by the vehicle over the time interval $[t_{k-1}, t_k]$ by

$$s(t_k) - s(t_{k-1}) \approx v(\tau_k)(t_k - t_{k-1}). \tag{2.11}$$

An estimate for $s(t)$, the distance traveled by the vehicle over the interval $[0,t]$, can then be estimated by

$$s(t) = \sum_{k=1}^{n} s(t_k) - s(t_{k-1}) \approx \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}), \tag{2.12}$$

where $t_o = 0$ and $s(0) = 0$. The approximations to $s(t)$ in (2.12) get better and better as n gets larger and larger. In fact, in the limit as n tends to infinity, we expect the sums on the right–most expression in (2.12) to tend to the $s(t)$, which is what we are trying to compute. We write

$$s(t) = \lim_{n\to\infty} \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}), \tag{2.13}$$

provided that the limit on the right–hand side of (2.13) exists. Later on in these notes we will see how to determine if that limit exists in general; we will also see how to compute those limits in some instances.

We will show how to compute the limit on the right–hand side of (2.13) for the case in which

$$\tau_k = k\frac{t}{n}, \qquad \text{for } k = 0, 1, 2, \ldots, n. \tag{2.14}$$

Using the definition of v in (2.6), (2.14) and (2.8) we can compute the sums on the right–hand side of (2.13) as follows:

$$\begin{aligned} \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) &= \sum_{k=1}^{n} ak\frac{t}{n} \cdot \frac{t}{n} \\ &= \frac{at^2}{n^2} \sum_{k=1}^{n} k. \end{aligned} \tag{2.15}$$

In Problem 4 of Assignment 1 you will be asked to verify that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{2.16}$$

Combine (2.15) and (2.16) to get

$$\sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) = \frac{at^2}{2}\frac{n+1}{n}, \tag{2.17}$$

which can be written as

$$\sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) = \frac{at^2}{2}\left(1 + \frac{1}{n}\right),$$

or

$$\sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) = \frac{at^2}{2} + \frac{at^2}{2n} \tag{2.18}$$

Observing that the last term in (2.18) tends to 0 as n goes to infinity; that is,

$$\lim_{n\to\infty} \frac{at^2}{2n} = 0, \tag{2.19}$$

we obtain from (2.18) and (2.19) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) = \frac{at^2}{2}. \tag{2.20}$$

Combining (2.13) and (2.20) then yields

$$s(t) = \frac{1}{2}at^2, \quad \text{for all } t \geqslant 0, \tag{2.21}$$

which yields a solution to the problem we set out to solve. The formula in (2.21) gives that distance traveled by the vehicle over the time interval $[0, t]$ for the case in which the vehicle's speed is given by (2.6), where a ia a constant of proportionality with units .

Remark 2.1.4. The argument outlined in the solution to the problem in Example 2.1.3 raises a lot of questions. For instance, will we get the same answer in (2.20) for any choice of τ_k's different from that in (2.14)? How do we know that we can make the assumption in (2.10)? We have already alluded to the fact that the answer to the second question is related to the notion of continuity that will be discussed later in these notes. We will also see that the assumption of continuity will allow as to give a positive answer to the first question as well. We will see this when we discuss the definite integral in these notes.

Remark 2.1.5. Another question we will need to address is that of the existence of the limits in (2.9) and (2.19). In these notes we will also spend some time developing the notion of limit, deriving its properties, and considering the various modes of limiting processes that come up in applications.

Remark 2.1.6. The limit expression

$$\lim_{n\to\infty} \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}),$$

provided that it exists for any choice of subdivisions of the interval $[0, t]$, is called the definite integral of v over the interval $[0, t]$ and is denoted by

$$\int_0^t v(\tau)\ d\tau.$$

Using this notation, the solution to the general problem of recovering the distance from the from the speed is

$$s(t) = \int_{t_o}^{t} v(\tau)\ d\tau, \quad \text{for all } t \geqslant t_o. \tag{2.22}$$

We will see when we study the theory of integration in more detail that, if R denotes the rate of change of a function f with respect to t, and R is continuous over an open interval containing $[t_o, t]$ then,

$$f(t) = f(t_o) + \int_{t_o}^{t} R(\tau)\ d\tau, \tag{2.23}$$

for all t in the interval of definition of f. Note that (2.22) is a special case of (2.23) for $f = s$, $R = v$, and $s(t_o) = 0$.

Chapter 3

The Concept of Limit

In the solution of the problem that we outlined in Chapter 2 we invoked the notion of a limit–the idea that quantity will tend to a some value as another variable tends to infinity or to certain value. For instance, in (2.19) we claimed that

$$\lim_{n\to\infty} \frac{at^2}{2n} = 0. \tag{3.1}$$

The idea behind (3.1) is that, as n gets larger and larger without bound, the numbers

$$\frac{at^2}{2}, \frac{at^2}{4}, \frac{at^2}{6}, \frac{at^2}{8}, \frac{at^2}{10}, \ldots, \tag{3.2}$$

for fixed values of a and t, will tend to 0.

In this chapter we will consider two notions of limit:

(i) the limit of a sequence of numbers, and

(ii) the limit of a function.

The underlying the concept of limit to be discussed in these notes is the notion of distance between two points on the real number line. If a and b are two real numbers, the distance from a to b is simply the absolute value of $a - b$,

$$|a - b|.$$

The absolute value of a real number, x, is defined by

$$|x| = \begin{cases} x & \text{if } x \geqslant 0; \\ -x & \text{if } x < 0. \end{cases}$$

A property of distance between real numbers that will be very useful in calculations of limits is the following fact known as the Triangle Inequality: For any real numbers a, b and c,

$$|a - b| \leqslant |a - c| + |c - b|. \tag{3.3}$$

3.1 Limits of Sequences

The infinite list of numbers in (3.2) is an example of an infinite sequence. In general, an infinite sequence

$$a_1, a_2, a_3, \ldots$$

is denoted by the symbol (a_n). Each a_n is a real number, and the index, n, indicates the place of the number in the list. For example, if $a_n = \frac{1}{n}$, then the symbol (a_n), or $\left(\frac{1}{n}\right)$, denotes the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

The notion of the limit of a sequence, (a_n), is the following. We say that a_n converges to a limit ℓ, is the distances $|a_n - \ell|$ tend to 0 as n goes to infinity. We write

$$\lim_{n\to\infty} |a_n - \ell| = 0, \tag{3.4}$$

or

$$\lim_{n\to\infty} a_n = \ell. \tag{3.5}$$

An intuitive meaning of the expressions in (3.4) is that the distances from the elements of the sequence in (a_n) to the limit ℓ can be made arbitrarily close to 0 by choosing sufficiently large indices n. This notion can be made very precise as is shown in Appendix B. However, we will rely on the notion described above along with a few facts about limits (many of which are proved in Appendix **??**) in order carry out calculations involving limits. We will list here several of those facts and show some examples on how to apply them in calculations.

Theorem 3.1.1 (Limit Fact 1)**.** The limit of the constant sequence (c) is c; that is,

$$\lim_{n\to\infty} c = c.$$

Remark 3.1.2. We can see the intuitive notion of limit in this section to see why the Limit Fact 1 is true. In the case of Limit Fact 1, $a_n = c$ for all indices n; thus, for all indices n,

$$|a_n - c| = |c - c| = 0,$$

so that the distance from each element of the sequence to c is 0.

Theorem 3.1.3 (Limit Fact 2)**.** The limit of the sequence $\left(\frac{1}{n}\right)$ is 0; that is,

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Remark 3.1.4. The proof of Limit Fact 2 relies on fundamental properties of the real numbers and is presented in Appendix B.

Theorem 3.1.5 (Squeeze Lemma)**.** Let (a_n), (b_n) and (c_n) be three sequences. Suppose that there exists a positive integer n_1 such that

$$a_n \leqslant b_n \leqslant c_n, \quad \text{for all } n \geqslant n_1.$$

Assume in addition that the sequences (a_n) and (c_n) converge to the same limit ℓ; that is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell.$$

Then, the sequence (b_n) converges to ℓ; that is,

$$\lim_{n\to\infty} b_n = \ell.$$

Remark 3.1.6. The proof of the Squeeze Lemma is presented in Appendix B.

Example 3.1.7. Use the Squeeze Lemma and the fact that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ to deduce that

$$\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Solution: Observe that

$$n+1 > n, \quad \text{for } n \geqslant 1,$$

from which we get that

$$\frac{1}{n+1} < \frac{1}{n}, \quad \text{for } n \geqslant 1,$$

so that

$$0 < \frac{1}{n+1} < \frac{1}{n}, \quad \text{for } n \geqslant 1. \tag{3.6}$$

Note that

$$\lim_{n\to\infty} 0 = 0 \tag{3.7}$$

by the Limit Fact 1 and

$$\lim_{n\to\infty} \frac{1}{n} = 0 \tag{3.8}$$

by the Limit Fact 2. It then follows from (3.6)–(3.8) and the Squeeze Lemma that

$$\lim_{n\to\infty} \frac{1}{n+1} = 0,$$

which was to be shown. □

Theorem 3.1.8 (Limit Fact 3)**.** Let (a_n) and (b_n) be sequences that converge to $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$. Then, the sequences $(a_n + b_n)$ and $(a_n b_n)$ converge and their limits are given by

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$, and

(ii) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

Example 3.1.9. Compute the limit of the sequence $\left(\dfrac{n+1}{n}\right)$.

Solution: Observe that

$$\frac{n+1}{n} = 1 + \frac{1}{n}, \qquad \text{for all } n \geqslant 1.$$

It then follows from Limit Facts 1 and 2, and from (i) in Limit Fact 3 that

$$\begin{aligned} \lim_{n\to\infty} \frac{n+1}{n} &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \\ &= \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n} \\ &= 1+0, \end{aligned}$$

so that

$$\lim_{n\to\infty}\frac{n+1}{n} = 1. \tag{3.9}$$

□

Theorem 3.1.10 (Limit Fact 4)**.** Let (a_n) and (b_n) be sequences that converge to $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$. Assume also that $\lim_{n\to\infty} b_n \neq 0$. Then, the sequences $\left(\dfrac{1}{b_n}\right)$ and $\left(\dfrac{a_n}{b_n}\right)$ converge and their limits are given by

(i) $\displaystyle\lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}$, and

(ii) $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Example 3.1.11. Compute the limit of the sequence $\left(\dfrac{n}{n+1}\right)$.

Solution: Observe that

$$\frac{n}{n+1} = \frac{1}{\dfrac{n+1}{n}}, \qquad \text{for all } n \geqslant 1.$$

It then follows from Limit Fact 1, (i) in Limit Fact 4 and the result of Example

3.1.13 in (3.9) that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{1}{\dfrac{n+1}{n}} \\
&= \frac{1}{\lim\limits_{n\to\infty} \dfrac{n+1}{n}} \\
&= \frac{1}{1},
\end{aligned}$$

so that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1. \tag{3.10}$$

□

Remark 3.1.12 (Alternate Solution to Problem in Example 3.1.13)**.** Alternatively, we can compute the limit in Example 3.1.13 as follows.

Divide numerator and denominator by n to get

$$\frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}}. \tag{3.11}$$

Next, take limits on both sides of (3.11) and use (ii) in Limit Fact 4 to compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n+1} &= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)} \\
&= \frac{1}{1+0} \\
&= 1,
\end{aligned}$$

where we have also used Limit Facts 1 and 2.

Example 3.1.13 (Example 2.1.3 Revisited)**.** In Example 2.1.3 we saw that, if a vehicle moves at a speed given by

$$v(t) = at, \quad \text{for all } t \geqslant 0, \tag{3.12}$$

where a is a positive constant measured in units of miles/hr^2 and t is measured in hours, then an approximation to the distance, $s(t)$, traveled by the vehicle over the interval $[0, t]$ is given by

$$s_n(t) = \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}) = \frac{at^2}{2}\frac{n+1}{n}, \tag{3.13}$$

(see (2.17) on page 11 in these notes). The expression in (3.13) defines a sequence, $(s_n(t))$, of approximations to the actual distance $s(t)$. The result of Example 3.1.9 shows that the the sequence $(s_n(t))$ has a limit given by

$$\lim_{n\to\infty} s_n(t) = \frac{at^2}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{at^2}{2}, \tag{3.14}$$

where we have also used Limit Facts 1 and 3.

Thus, it appears from (3.14) that a solution to the problem posed in Example 3.1.9 is

$$s(t) = \frac{at^2}{2}, \quad \text{for } t \geqslant 0. \tag{3.15}$$

In order to complete the solution of the problem posed in Example 3.1.9, we still need to answer the two questions posed in Remark 2.1.4 on page 12 in these notes. First, we need to see that the choice of subintervals

$$[0, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \quad \text{where } t_n = t, \tag{3.16}$$

and of times τ_k, for $k = 1, 2, \ldots, n$, within the subintervals in (3.16), respectively, will yield the same value in the limiting processes in (3.14), provided that the lengths of the subintervals in (3.16) tend to 0 as n tends to infinity. Even before we answer this question, we still need to make sense of the approximation

$$v(t) \approx v(\tau_k), \quad \text{for } t \in [t_{k-1}, t_k], \tag{3.17}$$

provided that the length of the subinterval is small.

The approximation in (3.17) will be justified by the continuity of the function v defined in (3.12). We will discuss continuity in the next chapter. Before we discuss continuity we need to develop the notion of limit of a function. This will be done in the next section.

3.2 Limits of Functions

Let f denote a real–valued function defined on some subset of the real number line. Let a denote a real number, which might or might not be in the domain of definition of f. We consider the question of what happens to the values, $f(t)$, of f as t tends to a.

Example 3.2.1. Let f be the real valued function defined by

$$f(t) = \frac{\sin(t)}{t}, \quad \text{for } t \neq 0. \tag{3.18}$$

Note that the function f given in (3.18) is not defined at 0. We would like to know what the values of f are doing as we take values of t (in radian measure) that get closer and closer to 0. Table 3.1 shows the values of $f(t)$ for values of t ranging from 0.10 to 0.04 in decrements of 0.01. We see from Table 3.1 that as the values of t decrease from 0.10 to 0.04 in steps of 0.01 (see first column of

t (radians)	$\sin(t)$	$\dfrac{\sin(t)}{t}$
0.10	0.09983	0.9983
0.09	0.08988	0.9987
0.08	0.07991	0.9989
0.07	0.06994	0.9991
0.06	0.05996	0.9993
0.05	0.04998	0.9996
0.04	0.03999	0.9998

Table 3.1: Some values of $\dfrac{\sin(t)}{t}$ for $t > 0$

Table 3.1), the values of $f(t)$ in the third column of the table increase towards 1. This suggests that the values of $\dfrac{\sin(t)}{t}$ tend 1 as the values t tend to 0. We write

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1. \tag{3.19}$$

Definition 3.2.2 (Informal Definition of Limit)**.** Let f denote a function whose domain is a subset of the real number line. Assume the domain of f is made up of intervals and that a is in one of those intervals or is an end–point of some of the intervals. (Note that a might or might not be in the domain of f). We say that f has a limit L at a, if the distance of the values $f(t)$ to L tends to 0 as the values of t tend towards a in the domain of f. We write

$$\lim_{t \to a} |f(t) - L| = 0,$$

or

$$\lim_{t \to a} f(t) = L.$$

Example 3.2.3. The function f given in Example 3.2.1 has domain given by the union of the intervals $(-\infty, 0)$ and $(0, -\infty)$. Note that 0 is an end–point of both intervals, but 0 is not in the domain of f. The calculations in Example 3.2.1 suggest that $\lim_{t \to 0} \dfrac{\sin(t)}{t}$ exists and equals 1; in other words, (3.19) is true. In subsequent examples in this section we will show that this is indeed the case.

The discussion in the remainder of this section will parallel the discussion of limits of sequences in Section 3.1. That is, we will state a few limit facts (most of which will be proved in Appendix B) and present examples that show how those limit facts can be used to compute limits. In Appendix B we give a precise definition of the limit of a function and present proofs of most of the limit facts.

Theorem 3.2.4 (Function Limit Fact 1)**.** Let $f(t) = c$ for all real numbers t, where c is a constant. Then, for any real number a,

$$\lim_{t\to a} f(t) = c.$$

Thus, the limit of a constant function, c, is c. We may also write

$$\lim_{t\to a} c = c.$$

Note that the statement in Function Limit Fact 1 makes sense according to the informal definition of limit in Definition 3.2.2 since, in this case, $f(t) = c$ for all t in $\mathbb{R}$; so that

$$|f(t) - c| = 0, \quad \text{for all } t \in \mathbb{R}.$$

Hence, the distance from $f(t)$ to c is already 0.

Theorem 3.2.5 (Function Limit Fact 2)**.** Let $f(t) = t$ for all real numbers t. Then, for any real number a,

$$\lim_{t\to a} f(t) = a.$$

We may also write

$$\lim_{t\to a} t = a.$$

Remark 3.2.6. To see why Function Limit Fact 2 is true according to Definition 3.2.2, note that, in this case,

$$|f(t) - a| = |t - a|;$$

so, we can make $|f(t) - a|$ go to 0 by making $|t - a|$ go to 0.

For future reference, we state Function Limit Fact 2 as follows:

$$\lim_{t\to 0} |t| = 0. \tag{3.20}$$

Theorem 3.2.7 (Function Limit Fact 3)**.** Let f and g be functions for which $\lim\limits_{t\to a} f(t)$ and $\lim\limits_{t\to a} g(t)$ exist. Then, the functions $f + g$ and fg have limits as t approaches a given by

(i) $\lim\limits_{t\to a}(f(t) + g(t)) = \lim\limits_{t\to a} f(t) + \lim\limits_{t\to a} g(t)$, and

(ii) $\lim\limits_{t\to a}(f(t)g(t)) = \lim\limits_{t\to a} f(t) \cdot \lim\limits_{t\to a} g(t)$.

Example 3.2.8. The function h given by $h(t) = t + 3$ has a limit as t tends to -2 given by

$$\lim_{t\to -2} h(t) = \lim_{t\to -2} t + \lim_{t\to -2} 3, \tag{3.21}$$

where we have used (i) in Function Limit Fact 3. Next, use Function Limit Facts 1 and 2 to get from (3.23) that

$$\lim_{t\to -2} h(t) = -2 + 3 = 1.$$

Example 3.2.9. Show that $\lim\limits_{t\to 0}\dfrac{t}{2}=0$.
Solution: Apply (ii) of Function Limit Fact 3 to get

$$\lim_{t\to 0}\frac{t}{2}=\lim_{t\to 0}\frac{1}{2}\cdot\lim_{t\to 0}t. \tag{3.22}$$

Next, apply Function Limit Fact 1 and (3.20) to get that

$$\lim_{t\to 0}\frac{t}{2}=\frac{1}{2}\cdot 0=0.$$

□

Theorem 3.2.10 (Function Limit Fact 4)**.** Let f and g be functions for which $\lim\limits_{t\to a}f(t)$ and $\lim\limits_{t\to a}g(t)$ exist. Assume also that $\lim\limits_{t\to a}g(t)\neq 0$. Then, the functions $\dfrac{1}{g}$ and $\dfrac{f}{g}$ have limits as t approaches a given by

(i) $\displaystyle\lim_{t\to a}\frac{1}{g(t)}=\frac{1}{\lim\limits_{t\to a}g(t)}$, and

(ii) $\displaystyle\lim_{t\to a}\frac{f(t)}{g(t)}=\frac{\lim\limits_{t\to a}f(t)}{\lim\limits_{t\to a}g(t)}$.

Theorem 3.2.11 (The Squeeze Lemma)**.** Let f, g and h denote a functions whose domains consist of union of intervals that either contain a, or a is an end–point of some of the intervals. (Note that a might or might not be in the domains of f, g or h). Suppose that there exists a positive number δ such that

$$f(t)\leqslant g(t)\leqslant h(t),\qquad \text{for } |t-a|<\delta,$$

and t is in the domains of f, g and h. Assume in addition that the limits of f and h as t approaches a exist and that

$$\lim_{t\to a}f(t)=\lim_{t\to a}h(t)=L.$$

Then, the limit of g as t approaches a exists and

$$\lim_{t\to a}g(t)=L.$$

Example 3.2.12. Let f be the real valued function defined by

$$f(t)=\frac{1-\cos t}{t},\qquad \text{for } t\neq 0. \tag{3.23}$$

We show that

$$\lim_{t\to 0}f(t)=0. \tag{3.24}$$

Solution: According to the informal definition of limit in Definition 3.2.2, in order to prove (3.24), it suffices to prove that

$$\lim_{t\to 0}\frac{|1-\cos t|}{|t|}=0, \tag{3.25}$$

in view of (3.23). We will use the Squeeze Lemma to prove (3.25).

We will need the Law of Cosines: □

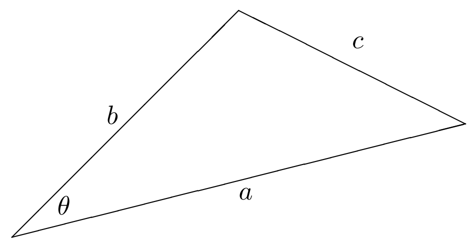

Figure 3.2.1: Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos(\theta), \tag{3.26}$$

where a, b and c are the lengths of the sides of a triangle pictured in Figure 3.2.1, and θ is the angle opposite the side of length c.

Consider a point, P, on the unit circle pictured in Figure 3.2.2. Suppose the

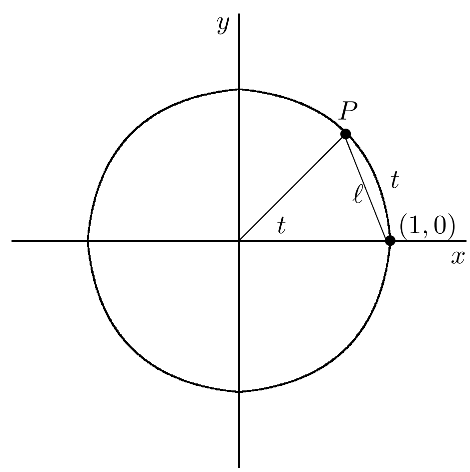

Figure 3.2.2: Unit Circle

point P is at a distance of $t > 0$ along the circular arc from $(1,0)$ to P. Let ℓ denote the distance from $(0,1)$ to P along the straight line segment connecting them. Applying the Law of Cosines (3.26) to the triangle in Figure 3.2.2 with vertices $(0,0)$, $(1,0)$ and P we obtain

$$\ell^2 = 1^2 + 1^2 - 2\cos t,$$

from which we get

$$1 - \cos t = \frac{\ell^2}{2} \tag{3.27}$$

Taking absolute values on both sides of (3.27) and using the fact that

$$|\ell| \leqslant |t|,$$

that is, the distance from P to $(1,0)$ along a straight line segment is shorter than that along any other path connecting the two points, we obtain from (3.27) that

$$|1 - \cos t| \leqslant \frac{|t|^2}{2}. \tag{3.28}$$

Assuming that $t \neq 0$ and dividing on both sides of (3.28) by $|t|$, we obtain from (3.28) that

$$0 < \frac{|1 - \cos t|}{|t|} < \frac{|t|}{2}, \quad \text{for } 0 < |t| < \frac{\pi}{2}. \tag{3.29}$$

The limit in (3.25) now follows from (3.29) by an application of the Squeeze Lemma since

$$\lim_{t \to 0} 0 = 0,$$

by Function Limit Fact 1, and

$$\lim_{t \to 0} \frac{|t|}{2} = 0,$$

by the result of Example 3.2.9.

Example 3.2.13. Compute $\lim\limits_{t \to 0} \cos t$.

Solution*:* We show that

$$\lim_{t \to 0} \cos t = 1, \tag{3.30}$$

or

$$\lim_{t \to 0} |\cos t - 1| = 0. \tag{3.31}$$

Observe that, for $t \neq 0$,

$$\begin{aligned} |\cos t - 1| &= |1 - \cos t| \\ &= |t| \cdot \frac{|1 - \cos t|}{|t|} \end{aligned} \tag{3.32}$$

Next, apply (ii) of Function Limit Fact 3 to the result of (3.32) to get

$$\begin{aligned}
\lim_{t\to 0} |\cos t - 1| &= \lim_{t\to 0} |t| \cdot \lim_{t\to 0} \frac{|1-\cos t|}{|t|} \\
&= 0 \cdot 0 \\
&= 0,
\end{aligned}$$

where we have also applied (3.20) and (3.25). We have therefore demonstrated (3.31). □

Example 3.2.14. Show that $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$.

Solution: Refer to the unit circle pictured in Figure 3.2.3. We first consider

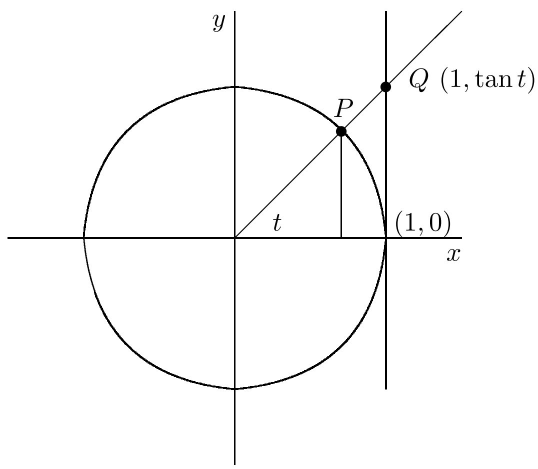

Figure 3.2.3: Comparing $\sin t$, t and $\tan t$.

the case in which a point P on the unit circle in the xy–plane has Cartesian coordinates $(\cos t, \sin t)$, where $0 < t < \dfrac{\pi}{2}$. Comparing the lengths of the vertical line segment from P to the x–axis, the arclength along the circle from P to $(1,0)$, and the length from Q to $(1,0)$ along the tangent line to the circle at $(1,0)$, we see from the sketch in Figure 3.2.3 that

$$\sin t < t < \tan t, \quad \text{for } 0 < t < \frac{\pi}{2},$$

from which we get that

$$\sin t < t < \frac{\sin t}{\cos t}, \quad \text{for } 0 < t < \frac{\pi}{2}. \tag{3.33}$$

Taking reciprocals in all the terms on the inequalities in (3.33) yields

$$\frac{\cos t}{\sin t} < \frac{1}{t} < \frac{1}{\sin t}, \qquad \text{for } 0 < t < \frac{\pi}{2}. \tag{3.34}$$

Multiplying all terms of the inequality in (3.34) by $\sin t$, for $0 < t < \dfrac{\pi}{2}$, we obtain that

$$\cos t < \frac{\sin t}{t} < 1, \qquad \text{for } 0 < t < \frac{\pi}{2}, \tag{3.35}$$

where we have used the fact that $\sin t > 0$ for $0 < t < \dfrac{\pi}{2}$.

Next, observe that each one of the functions in terms of the inequalities in (3.35) is even in their domains of definitions; that is,

$$\cos(-t) = \cos t, \qquad \text{for all } t$$

and

$$\frac{\sin(-t)}{-t} < \frac{\sin t}{t}, \qquad \text{for } t \neq 0,$$

we can say that

$$\cos t < \frac{\sin t}{t} < 1, \qquad \text{for } 0 < |t| < \frac{\pi}{2}. \tag{3.36}$$

Thus, in view of the result of Example 3.2.13 and Function Limit Fact 1, we obtain from (3.36) and the Squeeze Lemma that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1, \tag{3.37}$$

which we wanted to show. □

Example 3.2.15. Show that $\lim\limits_{t\to 0} \sin t = 0$.
***Solution**:* For $t \neq 0$, we can write

$$\sin t = t \cdot \frac{\sin t}{t}. \tag{3.38}$$

Taking limits on both sides of (3.38) and using (ii) in Function Limit Fact 3 and (3.37), we obtain from (3.38) that

$$\begin{aligned} \lim_{t\to 0} \sin t &= \lim_{t\to 0} t \cdot \lim_{t\to 0} \frac{\sin t}{t} \\ &= 0 \cdot 1 \\ &= 0, \end{aligned}$$

which was to be shown. □

Example 3.2.16. Show that $\lim_{t \to a} \sin t = \sin a$, for any real number a.
Solution: We show that

$$\lim_{t \to a} |\sin t - \sin a| = 0. \tag{3.39}$$

In order to show (3.39) we start with the following trigonometric identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{3.40}$$

and

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{3.41}$$

Subtracting (3.41) from (3.40) leads to

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta. \tag{3.42}$$

Next, put

$$\alpha + \beta = t \tag{3.43}$$

and

$$\alpha - \beta = a. \tag{3.44}$$

Solve (3.43) and (3.44) for α and β to get

$$\alpha = \frac{t + a}{2} \tag{3.45}$$

and

$$\beta = \frac{t - a}{2}. \tag{3.46}$$

Substitute (3.43) and (3.44) into the left–hand side of (3.42), and (3.45) and (3.46 into the right–hand side of (3.42) to get

$$\sin(t) - \sin(a) = 2\cos\left(\frac{t+a}{2}\right)\sin\left(\frac{t-a}{2}\right). \tag{3.47}$$

Next, take absolute values on both sides of (3.47) and use the estimates

$$|\cos\theta| \leqslant 1, \quad \text{for all } \theta,$$

and

$$|\sin\theta| \leqslant |\theta|, \quad \text{for } |\theta| < \frac{\pi}{2}$$

(see Problems 1 and 2 in Assignment #5), to obtain

$$|\sin(t) - \sin(a)| \leqslant |t - a|, \quad \text{for } |t - a| < \pi,$$

so that

$$0 \leqslant |\sin(t) - \sin(a)| \leqslant |t - a|, \quad \text{for } |t - a| < \pi. \tag{3.48}$$

Finally, use the facts that

$$\lim_{t \to a} 0 = 0 \quad \text{and} \quad \lim_{t \to a} |x - a| = 0$$

to get from (3.48) and the Squeeze Lemma that

$$\lim_{t \to a} |\sin t - \sin a| = 0,$$

which was to be shown. □

Chapter 4

The Concept of Continuity

We will see that the notion of limit discussed in the previous chapter is instrumental in defining the main concepts in Calculus: the integral and the derivative. In this chapter we see how limits can be used to define the concept of a continuous function. Continuity of the speed function in Example 2.1.3 was one of the crucial ingredients in the solution of the introductory problem that was outlined in that example.

4.1 Continuous Functions

In Example 3.2.16 we saw that

$$\lim_{t\to a} \sin t = \sin a.$$

In Problem 2 of Assignment #4 it is shown that, if p denotes a polynomial function defined by

$$p(t) = c_o + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_n t^n, \quad \text{for } t \in \mathbb{R},$$

where $c_o, c_1, c_2, \ldots, c_n$ are real constants, then

$$\lim_{t\to a} p(t) = p(a);$$

and in Problem 5 of Assignment #5 it is shown that

$$\lim_{t\to a} \cos t = \cos a.$$

A function, f, whose limit as t tends to a can be computed by evaluating the function at a,

$$\lim_{t\to a} f(t) = f(a), \tag{4.1}$$

is said to be continuous at a. We will use (4.1), or its equivalent form,

$$\lim_{t\to a} |f(t) - f(a)| = 0,$$

as the definition of continuity of f at a.

Definition 4.1.1 (Continuous Function)**.** Let f be a real–valued function defined in a domain containing a. We say that f is continuous at a if

$$\lim_{t\to a} |f(t) - f(a)| = 0. \tag{4.2}$$

If f is continuous at every point in its domain, we say that f is continuous on that domain.

Thus, according to (4.2), f is continuous at a if f is defined at a, and the values $f(t)$ can be made arbitrarily close to $f(a)$ by taking t in the domain of f sufficiently close to a.

Polynomial functions and the trigonometric functions sin and cos are examples of continuous functions on their entire domain of definition. In this section we learn how to identify more classes of continuous functions by learning a few properties of continuous functions.

Theorem 4.1.2 (Continuous Functions Fact 1)**.** Let f and g denote functions that are continuous at a; then, the functions $f + g$ and fg are also continuous at a. We then have that

$$\lim_{t\to a}(f + g)(t) = f(a) + g(a),$$

and

$$\lim_{t\to a}(fg)(t) = f(a) \cdot g(a).$$

Remark 4.1.3. Continuous Functions Fact 1 is a consequence of Function Limit Fact 3.

Example 4.1.4. The real–valued function f defined by

$$f(t) = t \sin t + (1 - t^2) \cos t, \qquad \text{for all } t \in \mathbb{R}$$

is continuous on $\mathbb{R}$ since it is a sum of products of continuous functions.

Theorem 4.1.5 (Continuous Functions Fact 2)**.** Let f and g denote functions that are continuous at a. Suppose $g(a) \neq 0$; then, the functions $\dfrac{1}{g}$ and $\dfrac{f}{g}$ are also continuous at a. Consequently, if f and g are continuous at a, and $g(a) \neq 0$, then

$$\lim_{t\to a} \frac{1}{g(t)} = \frac{1}{g(a)},$$

and

$$\lim_{t\to a} \frac{f(t)}{g(t)} = \frac{f(a)}{g(a)}.$$

Remark 4.1.6. Continuous Functions Fact 2 is a consequence of Function Limit Fact 4.

Example 4.1.7. The real–valued function f defined by

$$f(t) = \frac{\sin t}{t}, \quad \text{for } t \neq 0$$

is continuous at $a \neq 0$.

Definition 4.1.8 (Composition of Functions)**.** Let f g be two functions such that $g(t)$ lies in the domain of f for all t in the domain of g. We can then define the composition of f and g, denoted $f \circ g$, by

$$f \circ g(t) = f(g(t)), \quad \text{for } t \text{ in the domain of } g.$$

Example 4.1.9. The function h given by

$$h(t) = \sin(t^2), \quad \text{for all } t \in \mathbb{R},$$

is the composition of the trigonometric function sin and the polynomial function p given by $p(t) = t^2$ for all t in $\mathbb{R}$.

Theorem 4.1.10 (Continuous Functions Fact 3)**.** Let f and g be functions such that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a. We therefore have that

$$\lim_{t \to a} (f \circ g)(t) = f(g(a)).$$

Example 4.1.11. The function h given by

$$h(t) = \sin(t^2), \quad \text{for all } t \in \mathbb{R}.$$

Then, h is continuous at every a in $\mathbb{R}$; so that,

$$\lim_{t \to a} \sin(t^2) = \sin(a^2),$$

for all a is $\mathbb{R}$.

Example 4.1.12. The function g given by

$$g(t) = (\sin t)^2, \quad \text{for all } t \in \mathbb{R}.$$

is the composition of the polynomial function, p, given by

$$p(u) = u^2, \quad \text{for all } u \in \mathbb{R},$$

the trigonometric function sin. In fact,

$$g(t) = p \circ \sin(t) = p(\sin t), \quad \text{for all } t \in \mathbb{R}.$$

Since, p and sin are continuous on $\mathbb{R}$, it follows that g is continuous on $\mathbb{R}$ and

$$\lim_{t \to a} (\sin t)^2 = (\sin a)^2 = \sin^2 a,$$

for all a is $\mathbb{R}$.

4.2 Discontinuous Functions

We begin this section with an example.

Example 4.2.1. We saw in Example 4.1.7 that the real–valued function f defined by

$$f(t) = \frac{\sin t}{t}, \qquad \text{for } t \neq 0, \tag{4.3}$$

is continuous everywhere except at $a = 0$. However, in Example 3.2.14 we saw that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1. \tag{4.4}$$

The limit fact in (4.4) suggests that we can define a new function, which we will denote by $\widehat{f}$, which agrees with f for $t \neq 0$, and is 1 at $t = 0$:

$$\widehat{f}(t) = \begin{cases} \dfrac{\sin t}{t}, & \text{if } t \neq 0; \\ \\ 1, & \text{if } t = 0. \end{cases} \tag{4.5}$$

The new function $\widehat{f}$ defined in (4.5) is continuous at 0 (in fact, it is continuous at all points in $\mathbb{R}$) and it agrees with f for $t \neq 0$. We say that $\widehat{f}$ removes the discontinuity of f at 0; or that f has a **removable discontinuity** at 0. Figure 4.2.1 shows a sketch of the graph of $y = \widehat{f}(t)$.

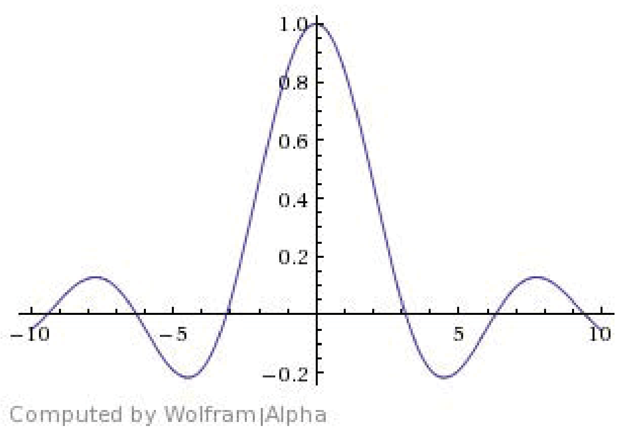

Figure 4.2.1: Sketch of graph of $y = \widehat{f}$

Definition 4.2.2 (Removable Discontinuity)**.** A function f, which is not defined at a, is said to have a **removable discontinuity** at a if $\lim\limits_{t \to a} f(t)$ exists. In order to remove the discontinuity of f at a, we define f at a to be $\lim\limits_{t\to a} f(t)$; that is,

$$f(a) = \lim_{t\to a} f(t).$$

Example 4.2.3. The function f defined by

$$f(t) = t \cdot \sin\left(\frac{1}{t}\right), \qquad \text{for } t \neq 0, \tag{4.6}$$

is discontinuous at 0 since the argument of the sine function in (4.6) is not defined at 0. However, we will see in this example that $\lim\limits_{t\to 0} f(t)$ exists; in fact, we will see that

$$\lim_{t\to 0}\left[t \cdot \sin\left(\frac{1}{t}\right)\right] = 0. \tag{4.7}$$

To see why (4.7) is true, take absolute in (4.6) to obtain

$$|f(t)| = |t|\left|\sin\left(\frac{1}{t}\right)\right|, \qquad \text{for } t \neq 0. \tag{4.8}$$

Then, use the inequality

$$|\sin\theta| \leqslant 1, \qquad \text{for all } \theta \in \mathbb{R},$$

to obtain from (4.8) that

$$0 \leqslant |f(t)| \leqslant |t|, \qquad \text{for } t \neq 0. \tag{4.9}$$

Thus, by the Squeeze Lemma, it follows from (4.9) that

$$\lim_{t\to 0} f(t) = 0, \tag{4.10}$$

which is (4.7).

It follows from (4.10) that f has a removable discontinuity at 0, an so f can be defined at 0 to make it continuous everywhere:

$$f(t) = \begin{cases} t \cdot \sin\left(\dfrac{1}{t}\right), & \text{if } t \neq 0; \\ \\ 0, & \text{if } t = 0. \end{cases} \tag{4.11}$$

A sketch of the graph of f is shown in Figure 4.2.2.

Thus, if f is not continuous at a, but $\lim\limits_{t\to a} f(t)$ exists, f can be defined at a so as to make f continuous at a. However, if $\lim\limits_{t\to a} f(t)$ does not exist, the discontinuity at a cannot be removed. There are several ways in which the limit $\lim\limits_{t\to a} f(t)$ might fail to exist. We present here three examples that illustrate how this might happen.

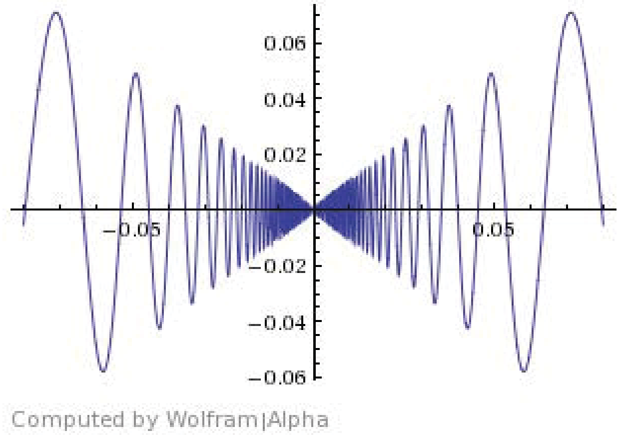

Figure 4.2.2: Sketch of graph of f defined in (4.11)

Example 4.2.4. Let f denote the real–valued function defined by

$$f(t) = \frac{\sin t}{|t|}, \quad \text{for } t \neq 0. \tag{4.12}$$

Note that f is not continuous at $a = 0$. In order to study the nature of the discontinuity of f at 0, consider a sketch of the graph of $y = f(t)$, for $t \neq 0$ shown in Figure Observe from the sketch in Figure 4.2.3 that the value $f(t)$

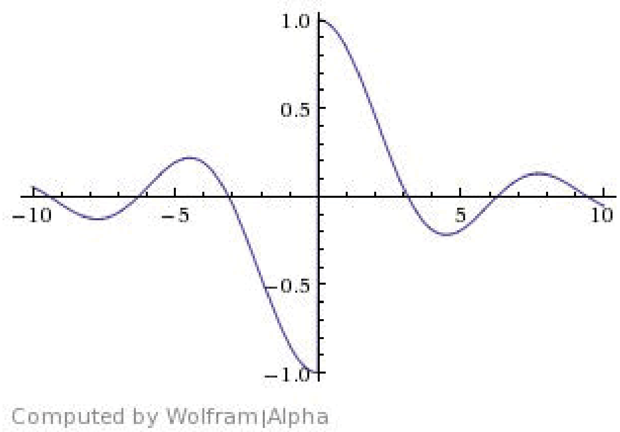

Figure 4.2.3: Sketch of graph of f defined in (4.12)

appear to be approaching $+1$ as t approaches 0 along positive values, and it appears to approach -1 as t approaches 0 along negative values. We express these facts in symbols as follows

$$\lim_{t\to 0^+} f(t) = 1, \tag{4.13}$$

and

$$\lim_{t\to 0^-} f(t) = -1, \tag{4.14}$$

The expressions in (4.13) and (4.14) are known as **one–sided limits**.

Thus, the one–sided limits of f as t approaches 0 exist; but $\lim\limits_{t\to 0} f(t)$ does not exist. The reason that the limit of the function f defined in (4.12) as t approaches 0 does not exist follows from the uniqueness of limits shown in Appendix B. In other words, if a limit exists as t approaches a given number, it can be at most one value. For the function in (4.12), $f(t)$ approaches 1 and t approaches 0 through positive values, and $f(t)$ approaches -1 as t approaches 0 through negative values; $+1$ and -1 are two distinct values. In this case, we say that f has a **jump discontinuity** at $a = 0$. This type of discontinuity cannot be removed.

Definition 4.2.5 (Jump Discontinuity)**.** A function f, which is not defined at a, is said to have a **jump discontinuity** at a if the one–sided limits $\lim\limits_{t\to a^+} f(t)$ and $\lim\limits_{t\to a^-} f(t)$ exist, but

$$\lim_{t\to a^+} f(t) \neq \lim_{t\to a^-} f(t).$$

Example 4.2.6. The function f defined by

$$f(t) = \begin{cases} 0, & \text{if } t < 0; \\ -2t, & \text{if } 0 \leqslant t < \frac{1}{2}; \\ 4(1-t), & \text{if } \frac{1}{2} \leqslant t < 1 \\ 0 & \text{if } t \geqslant 1, \end{cases} \tag{4.15}$$

is continuous everywhere except at $\dfrac{1}{2}$, where f has a jump discontinuity. To see why this assertion is true, since f is defined in a piecewise manner, we need to compute the one–sided limits at the endpoints of the intervals

$$(-\infty, 0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \quad \text{and} \quad (1, +\infty). \tag{4.16}$$

To see that f is continuous on each of the open intervals in (4.16), we note that, according to (4.15), f is either a constant or a polynomial; thus, f is continuous on those intervals. It remains, then, to see what happens at the endpoints of the intervals in (4.16).

At 0, we compute the one–sided limits

$$\lim_{t\to 0^-} f(t) = \lim_{t\to 0^-} 0 = 0, \tag{4.17}$$

and

$$\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} (-2t) = 0. \tag{4.18}$$

Since the two one–sided limits in (4.17) and (4.18) are equal, it follows that

$$\lim_{t\to 0} f(t) = 0 = f(0),$$

which shows that f is continuous at 0. Similar calculations at 1; namely,

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 0^-} 4(1-t) = 0$$

and

$$\lim_{t\to 1^+} f(t) = \lim_{t\to 0^-} 0 = 0,$$

which show that

$$\lim_{t\to 1} f(t) = 0 = f(1);$$

so that f is continuous at 1.

On the other hand, at $\dfrac{1}{2}$ we obtain that

$$\lim_{t\to \frac{1}{2}^-} f(t) = \lim_{t\to \frac{1}{2}^-} (-2t) = -1, \tag{4.19}$$

while

$$\lim_{t\to \frac{1}{2}^+} f(t) = \lim_{t\to \frac{1}{2}^+} 4(1-t) = 2. \tag{4.20}$$

Since the one–sided limits in (4.19) and (4.20), are not equal, it follows that f has a jump discontinuity at $\dfrac{1}{2}$. The values of the one–sided limits indicate that the values of $f(t)$ "jump" from -1 to 2 (a jump of 3 units), hence the name of the discontinuity.

Figure 4.2.4 shows a sketch of the graph of f defined in (4.15).

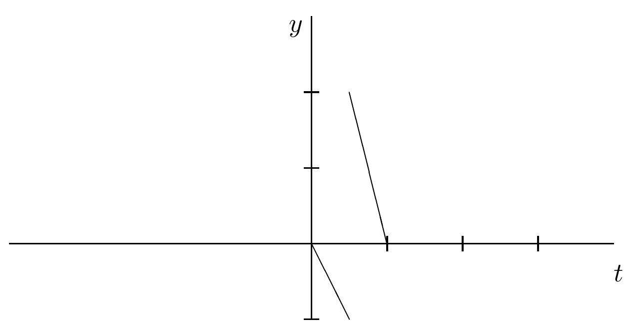

Figure 4.2.4: Sketch of graph of f in (4.15).

The function in Example 4.2.6 is an example of a piecewise continuous function.

Definition 4.2.7 (Piecewise Continuous Function)**.** A function f defined on a closed and bounded interval, $[a, b]$, is said to be piecewise continuous if

$$\lim_{t\to a^+} f(t) = f(a),$$

$$\lim_{t\to b^-} f(t) = f(b),$$

and f is continuous on (a, b) except for a finite number of jump discontinuities or removable discontinuities.

Example 4.2.8. Consider the function f defined by

$$f(t) = \frac{1}{t}, \qquad \text{for } t \neq 0. \tag{4.21}$$

Note that f defined in (4.21) is continuous everywhere except at 0 because it is the ratio of two continuous functions with the denominator not equal to 0 for $t \neq 0$. In this case the one–sided limits as t approaches 0 do not exist; in fact,

$$\lim_{t\to 0^+} \frac{1}{t} = +\infty, \tag{4.22}$$

and

$$\lim_{t\to 0^-} \frac{1}{t} = -\infty. \tag{4.23}$$

The meaning of the expressions in (4.22) and (4.23) can be made precise (this is done in Appendix B); however, they can be interpreted as follows: The values of $\dfrac{1}{t}$ increase without bound as t tends to 0 through positive values; so that, no limiting value is attained; similarly, the values of $\dfrac{1}{t}$ decrease without bound as t tends to 0 through negative values, so that that not limiting value is attained in this case either. Geometrically, (4.22) and (4.23) imply that the graph of $y = \dfrac{1}{t}$, for $t \neq 0$, has a vertical asymptote at $t = 0$; i.e, the y–axis is a vertical asymptote to the graph of $y = \dfrac{1}{t}$, for $t \neq 0$, sketched in Figure 4.2.5. For this reason, a discontinuity of the time that $f(t) = \dfrac{1}{t}$, for $t \neq 0$, has $t = 0$ is called the **vertical asymptote discontinuity**.

Definition 4.2.9 (Vertical Asymptote Discontinuity)**.** A function f, which is not defined at a, is said to have a **vertical asymptote discontinuity** at a if either

$$\lim_{t\to a^+} f(t) = \pm\infty,$$

or

$$\lim_{t\to a^-} f(t) = \pm\infty.$$

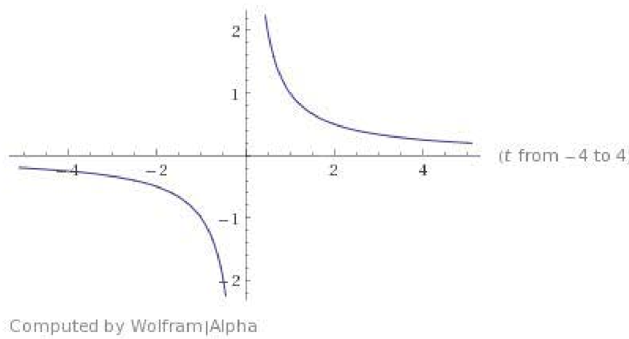

Figure 4.2.5: Sketch of graph of f defined in (4.21)

Example 4.2.10. The tangent function, tan, given by

$$\tan(t) = \frac{\sin t}{\cos t}, \qquad \text{for } t \neq (2k+1)\frac{\pi}{2}, \tag{4.24}$$

for any integer k, is continuous everywhere except at odd multiples of $\frac{\pi}{2}$, at which the graph of $y = \tan t$ has vertical asymptotes (see sketch in Figure 4.2.6).

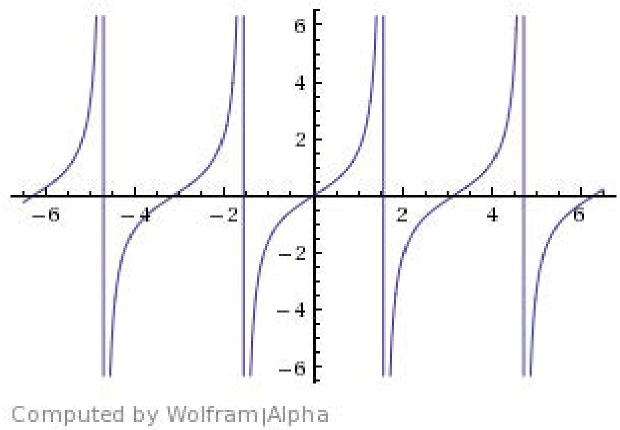

Figure 4.2.6: Sketch of graph of $y = \tan t$

In the previous examples the limit of f as t approaches a fails to exist because either the one–sided limits at a are not equal, or $|f(t)|$ grows without bound as t approaches a from the left or from the right. In the next example we see a situation in which the values of $f(t)$ remain bounded in a neighborhood of a, and the one–sided limits fail to exist.

Example 4.2.11. Let f be given by

$$f(t) = \sin\left(\frac{1}{t}\right), \qquad \text{for } t \neq 0. \tag{4.25}$$

An examination of the graph of $y = f(t)$, obtained using WolframAlpha®, in Figure 4.2.7 reveals that the values of $f(t)$, where f is as defined in (4.25)

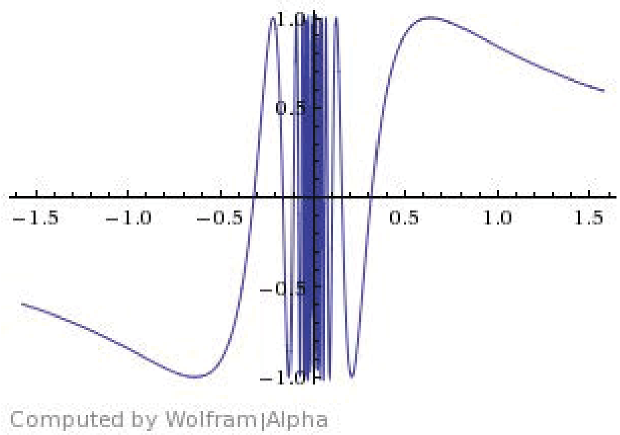

Figure 4.2.7: Sketch of graph of $y = \sin(1/t)$ for $t \neq 0$

appears to be approaching any value real in the y–axis between -1 and 1 as t approaches 0; thus, no limiting value for $f(t)$ can be attained in this case, since, as shown in Appendix B, if f has a limit at 0, it can be at most one value.

In these notes, we shall refer to a discontinuity of the type illustrated in Example 4.2.11 as an **essential discontinuity**. In an essential discontinuity, the values of $f(t)$ appear to be approaching a range of values as t approaches the point of discontinuity. In the case of the function in Example 4.2.11, this assertion can be made precise by the use of the following fact about the relation between limits of functions and limits of sequences:

Theorem 4.2.12 (Functional Limits and Sequential Limits Fact)**.** A real valued function f has a limit, L, as t approaches a if an only if, for every sequence of real numbers, (t_n), that converges to a,

$$\lim_{n\to\infty} f(t_n) = L.$$

Theorem 4.2.12 is discussed in Appendix B. We will have occasions to apply this theorem later on in these notes. For the case of the function f defined in (4.25), we can see that the limit of f as t approaches 0 does not exist since we can come up with a range of values; namely,

$$\sin\theta, \qquad \text{for } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2},$$

and a range of sequences; namely, (t_n) with

$$t_n = \frac{1}{\theta + 2\pi n}, \qquad \text{for } n = 1, 2, 3, \ldots,$$

such that

$$\lim_{n\to\infty} t_n = 0,$$

and

$$\lim_{n\to\infty} \sin\left(\frac{1}{t_n}\right) = \sin\theta. \tag{4.26}$$

Thus, according to Theorem 4.2.12, the limit of $\sin\left(\frac{1}{t}\right)$ as t approaches 0 cannot exist since θ in (4.26) can be chosen to be any value in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4.3 Properties of Continuous Functions

In this section we list a few properties of continuous functions that are very useful in applications. We will have several occasions in these notes to use these properties. Proofs of these properties may be found in Appendix **??**. The first of these properties is known and the Intermediate Value Theorem. It is the property behind the notion that the graph of a continuous function can be drawn or sketched without having to lift the pencil from the paper. The second property states that a continuous function attains its maximum or minimum values on closed and bounded intervals. This last properties is very useful when finding maxima or minima of quantities that may be assumed to be continuous (optimization problems).

Theorem 4.3.1 (The Intermediate Value Property)**.** Suppose that f is continuous on an open interval that contains the closed interval $[a, b]$. Suppose that $f(a) < f(b)$ or $f(a) > f(b)$; then, for any value y between $f(a)$ and $f(b)$, there exists a point c in the interval (a, b) for which

$$f(c) = y. \tag{4.27}$$

Remark 4.3.2. Theorem 4.3.1 is useful when one is trying to see whether the equation

$$f(x) = y \tag{4.28}$$

has a real solution for a given real number y. It is it known that f is continuous on some interval containing the points a and b, with $a < b$, and it is also known, or instance, that $f(a) < y < f(b)$, then, according to (4.27), the Intermediate Value Property of continuous functions guarantees that the equation in (4.28) has at least one solution $x = c$ in in the interval (a, b). Thus, the Intermediate Value Property can be used to locate solutions.

Example 4.3.3. In this example we show that any cubic polynomial of the form $p(t) = t^3 + bt^2 + ct + d$, where a, b, c and d are real constants, has at least one real root; that is, there exists at least one real solution to the cubic polynomial equation

$$t^3 + bt^2 + ct + d = 0,$$

or

$$p(t) = 0. \tag{4.29}$$

Since p is continuous on $\mathbb{R}$, we can show that (4.29) has at least one real solution by applying the Intermediate Value Property of continuous functions. In order to do this, we will show that there exists a positive real number R such that

$$p(-R) < 0 < p(R). \tag{4.30}$$

Then, by Theorem 4.3.1, there exists c in the interval $(-R, R)$ such

$$p(c) = 0.$$

In order to establish (4.30), first assume that $t \neq 0$ and divide $p(t)$ by t^3 to get that

$$\frac{p(t)}{t^3} = 1 + \frac{b}{t} + \frac{c}{t^2} + \frac{d}{t^3}, \qquad \text{for } t \neq 0. \tag{4.31}$$

Next, assume that $|t| \geqslant 1$ and use the inequalities $|t| \leqslant |t|^2$ and $|t| \leqslant |t|^3$, for $|t| \geqslant 1$, to obtain from (4.31) and the triangle inequality that

$$\frac{p(t)}{t^3} \geqslant 1 - \frac{1}{|t|}[|b| + |c| + |d|], \qquad \text{for } |t| \geqslant 1. \tag{4.32}$$

Next, choose R to be the larger of 1 and $\dfrac{2}{|b| + |c| + |d| + 1}$. It then follows from (4.32) that

$$\frac{p(t)}{t^3} > \frac{1}{2}, \qquad \text{for } |t| \geqslant R. \tag{4.33}$$

We then get from (4.33) that

$$p(R) > \frac{R^3}{2} > 0 \quad \text{ and } \quad p(-R) < -\frac{R^3}{2} < 0,$$

which is (4.30).

Definition 4.3.4 (Bounded Functions)**.** We say that a real valued function, f, is bounded over some subset, A, of the domain of f if there exists a positive constant M such that

$$|f(t)| \leqslant M, \qquad \text{for all } t \in A.$$

Example 4.3.5. The function f defined by $f(t) = \dfrac{1}{t}$, for $t \neq 0$, is bounded over the interval $[1, 2]$ since

$$|f(t)| \leqslant 1, \quad \text{for } 1 \leqslant t \leqslant 2.$$

However, f is not bounded over $(0, 1)$ since

$$\lim_{t \to 0^+} \frac{1}{t} = +\infty.$$

The next theorem states that continuous function defined over a closed and bounded interval must be bounded.

Theorem 4.3.6 (Boundedness Property of Continuous Functions)**.** Suppose that f is continuous on the closed interval $[a, b]$, then there exists a real value $M > 0$ such that

$$|f(t)| \leq M \qquad \text{for all } t \text{ in the interval } [a, b].$$

That is, f is bounded on $[a, b]$.

A continuous function not only is bounded over a closed and bounded interval, but it also attains its maximum and minimum values in that interval; more precisely,

Theorem 4.3.7 (Maximum Property of Continuous Functions)**.** Suppose that f is continuous on the closed interval $[a, b]$, then there exists t_1 in $[a, b]$ such that

$$f(t) \leqslant f(t_1) \quad \text{for all } \ t \ \text{ in the interval } \ [a, b].$$

That is, f takes on its maximum value on $[a, b]$ at a point t_1 in $[a, b]$.

Theorem 4.3.8 (Minimum Property of Continuous Functions)**.** Suppose that f is continuous on the closed interval $[a, b]$, then there exists t_2 in $[a, b]$ such that

$$f(t) \geqslant f(t_2) \quad \text{for all } \ t \ \text{ in the interval } \ [a, b].$$

That is, f takes on its minimum value on $[a, b]$ at a point t_2 in $[a, b]$.

The boundedness property is also true for piecewise continuous functions (see Definition 4.2.7 on page 37 in these notes).

Theorem 4.3.9 (Boundedness Property of Piecewise Continuous Functions)**.** Suppose that f is piecewise continuous on the closed interval $[a, b]$, then there exists a real value $M > 0$ such that

$$|f(t)| \leq M \qquad \text{for all } t \text{ in the interval } [a, b].$$

That is, f is bounded on $[a, b]$.

Chapter 5

Integral Calculus

In Chapter 2 we alluded to the fact that the notion of the integral of a function has to do with the solution to the problem that we presented in that chapter; namely, the problem of recovering a function from its rate of change. We begin this chapter with a geometric problem, which on a first encounter does not seem to be related to the problem discussed in Chapter 2: the problem of computing areas of bounded plane regions.

5.1 The Area Problem

In this section we discuss a geometric problem that goes back to antiquity: determining the area of a bounded plane region.

Problem 5.1.1 (Area Problem)**.** Given a bounded region, R, in the xy–plane, if possible, compute the area of R; denoted area(R).

If the region R is a polygonal region, like the one shown in Figure 5.1.1, then the area problem is not hard to solve. For instance, we can divide the region into triangles, as shown in Figure 5.1.2 We can then use the formula for computing the area of a triangle, T, with base of length b and height h:

$$\text{area}(T) = \frac{1}{2}bh. \tag{5.1}$$

(See the triangle in Figure 5.1.3 to see the meanings of b and h in the formula in (5.1)). We can then obtain a formula for the area of the region R by adding the areas of all the triangles that make up R. More explicitly, suppose that there are n triangles that make up the region R; denote them by $T_1, T_2, \ldots, T_n$. Then,

$$\text{area}(R) = \sum_{k=1}^{n} \text{area}(T_k).$$

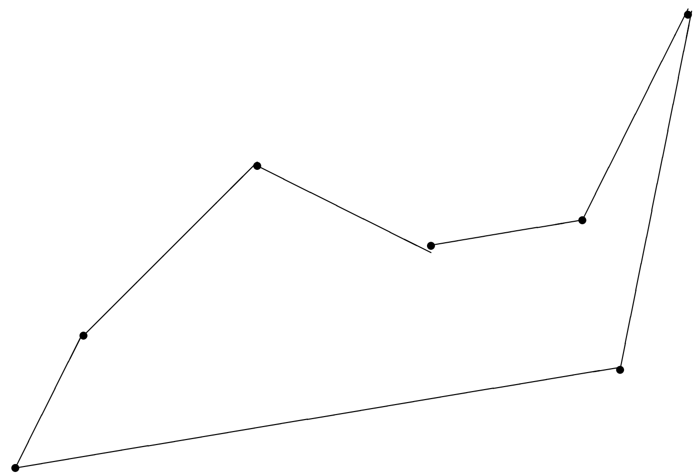

Figure 5.1.1: Polygonal Plane Region

Example 5.1.2. Compute the area of the octagon inscribed in the circle of radius r shown in Figure 5.1.4.

Solution: The octagon can be divided into eight congruent triangles as shown in Figure 5.1.5. Thus, the area of the octagon, R, shown in Figures 5.1.4 and 5.1.5 is given by

$$\text{area}(R) = 8 \cdot \text{area}(T), \tag{5.2}$$

where T is an isosceles triangle with equal sides of length r and angle, θ, between the two equal sides given by

$$\theta = \frac{2\pi}{8} = \frac{\pi}{4}. \tag{5.3}$$

We use the formula in (5.1) to compute the area of the triangle T pictured in Figure 5.1.6. The dashed line segment in Figure 5.1.6 is the perpendicular bisector of the base of the triangle since T is isosceles; it is also the angular bisector to the angle θ in (5.3). Its length is therefore h, the height of the triangle. Using the definition of the trigonometric functions sin and cos, we obtain from the sketch in Figure 5.1.6 that

$$h = r\cos\left(\frac{\theta}{2}\right), \tag{5.4}$$

and

$$\frac{b}{2} = r\sin\left(\frac{\theta}{2}\right),$$

or

$$b = 2r\sin\left(\frac{\theta}{2}\right). \tag{5.5}$$

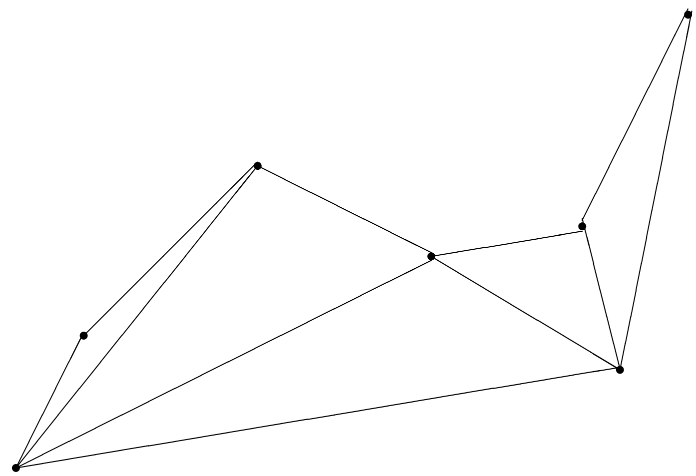

Figure 5.1.2: Polygonal Plane Region Divided into Triangles

Combining (5.1), (5.4) and (5.5) we obtain that the area of the triangle T in Figure 5.1.6 is

$$\text{area}(T) = \frac{1}{2}r^2 \left[2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]. \tag{5.6}$$

Next, use the trigonometric identity

$$\sin(2\beta) = 2\sin\beta\cos\beta$$

to get from (5.6) that

$$\text{area}(T) = \frac{1}{2}r^2\sin\theta. \tag{5.7}$$

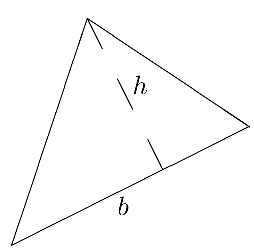

Figure 5.1.3: Triangle T of base of length b and height h

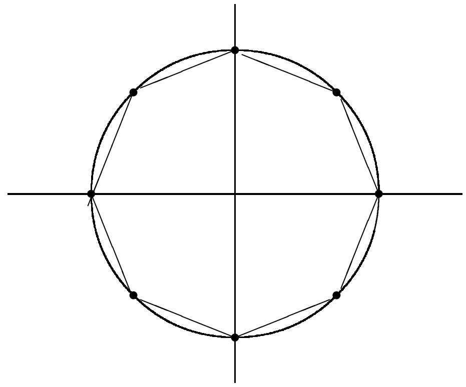

Figure 5.1.4: Inscribed Octagon

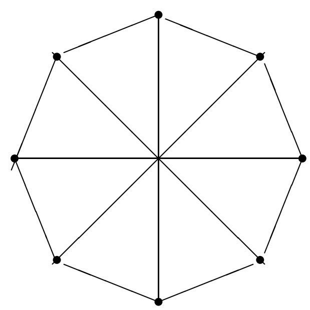

Figure 5.1.5: Octagon Divided into Eight Congruent Triangles

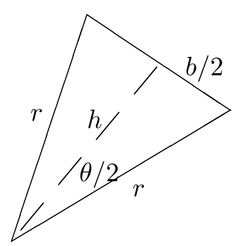

Figure 5.1.6: Isosceles Triangle T

We will now use the formula in (5.7) and (5.2) to compute the area of a regular octagon inscribed in a circle of radius r:

$$\text{area}(R) = 4r^2 \sin\theta, \tag{5.8}$$

where θ is given in (5.3). We then have from (5.3) and (5.8) that

$$\text{area}(R) = 4r^2 \frac{\sqrt{2}}{2} = 2\sqrt{2}\, r^2.$$

□

We will be using the formula in (5.7) derived in Example 5.1.2 one more time in these notes. The formula gives the area of an isosceles triangle with equal sides of length r and angle θ between them in terms of r and the sine of θ.

Ideally, we would like to compute the area of bounded regions in the plane whose boundaries are not necessarily polygons; for instance, the area of the elliptical region pictured in Figure 5.1.7. For the case of a region R bounded

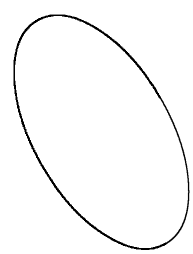

Figure 5.1.7: Elliptical Region

by an ellipse as in Figure 5.1.7, division into triangular regions is not possible because there are no straight segments on the boundary. However, we can use the same idea that was used in the solution to the problem in Chapter 2: use approximations and limits. For instance, we can approximate the area of the region by the area of inscribed polygons (since we already know how to compute those areas) and then consider the process of refining the approximations by using polygons with large numbers of sides with lengths that are very small. The idea here is that, in the limit as the number of sides goes to infinity, the boundary of the polygon will approach the boundary of the region. We will illustrate the use of this idea, which is at the core of the Integral Calculus, by computing the area of the region enclosed by the circle of radius r and centered at the origin shown in Figure 5.1.8.

Example 5.1.3. Compute the area of the circle of radius r shown in Figure 5.1.8.

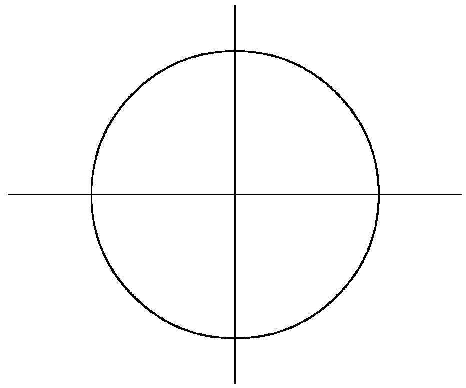

Figure 5.1.8: Circle of Radius r centered at the origin

Solution: Inscribe a regular polygon of n sides in the circular region in Figure 5.1.8 as shown in Figure 5.1.9. (Note: Figure 5.1.9 depicts the case $n = 8$).

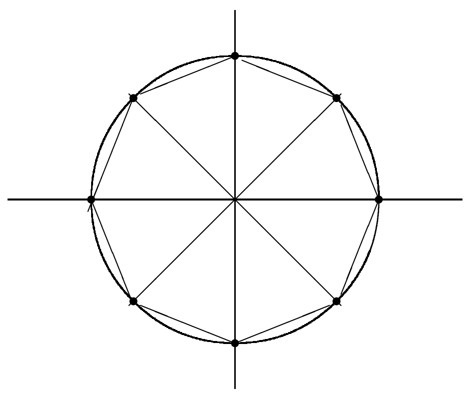

Figure 5.1.9: Circle of Radius r with inscribed regular n–gon

Denote the region bounded by the inscribed n–gon by P_n and by T any of the triangular regions making up P_n in Figure 5.1.9. Then,

$$\text{area}(P_n) = n \cdot \text{area}(T), \tag{5.9}$$

where, according to the formula in (5.7),

$$\text{area}(T) = \frac{1}{2} r^2 \sin\theta, \quad \text{where } \theta = \frac{2\pi}{n}. \tag{5.10}$$

We then obtain from (5.9) and (5.10) that

$$\text{area}(P_n) = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right),$$

which can be re–written as

$$\text{area}(P_n) = \pi r^2 \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right), \quad \text{for } n = 3, 4, 5, \ldots \tag{5.11}$$

Set

$$t_n = \frac{2\pi}{n}, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.12}$$

Then, (5.12) defines a sequence (t_n) that converges to 0; that is

$$\lim_{n\to\infty} t_n = 0. \tag{5.13}$$

Using (5.12) we can re–write (5.11) as

$$\text{area}(P_n) = \pi r^2 \frac{\sin(t_n)}{t_n}, \quad \text{for } n = 3, 4, 5, \ldots \tag{5.14}$$

Recalling that

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1,$$

(see (3.37) in Example 3.2.14), we obtain from (5.14), (5.13) and Theorem 4.2.12 on page 39 in these notes that

$$\lim_{n\to\infty} \text{area}(P_n) = \pi r^2 \lim_{n\to\infty} \frac{\sin(t_n)}{t_n} = \pi r^2 \cdot (1) = \pi r^2, \tag{5.15}$$

which gives a formula for computing the area of a region enclosed by a circle of radius r. □

Example 5.1.3 illustrates two fundamental notions in the Integral Calculus. First, there is the notion of approximating a quantity by a sum of parts of the whole we are trying to compute (in this case, areas of component triangles that comprise the region whose area we are trying to compute). Secondly, there is a limiting process that takes place as the number of component pieces increases to infinity as the dimensions associated with the component parts tend to 0 (in Example 5.1.3, the angles, t_n, associated with each triangle making up the approximating polygon tend to 0 as the number of sided of the polygon increases to infinity). In subsequent sections we will see how these two notions of approximating sums and limiting processes can be used to compute areas of regions in the plane bounded by graphs of piece–wise continuous functions.

5.2 The Area Function

Continuing with the theme of computing areas of plane regions, in this section we consider the case regions in the ty–plane bounded by the graph of a piecewise continuous function, f, the t–axis, and a pair of vertical lines. Figure 5.2.10 illustrates the situation for a nonnegative, piecewise continuous function. We

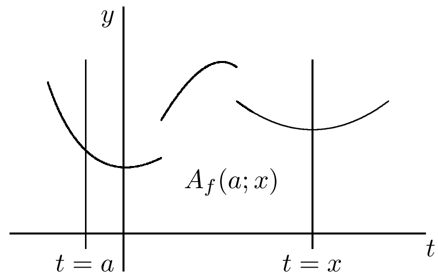

Figure 5.2.10: $A_f(a;x)$

are interested in computing the area of the region in the ty–plane that lies below the graph of $y = f(t)$, above the t–axis and between the vertical lines $t = a$ and $t = x$, where we are assuming that $a < x$. We will denote the area of that region by $A_f(a;x)$. We imagine that we can compute $A_f(a;x)$ for various values of x as x moves along the t–axis. We therefore obtain a function of x, which we shall call the area function of f from a to x. We begin with the simplest example: the constant function $f(t) = c$, for all $t \in \mathbb{R}$, where $c > 0$.

Example 5.2.1. Let $f(t) = c$ for all t, where $c > 0$. Let a denote a point on the t–axis as depicted in Figure 5.2.11. Compute $A_f(a;x)$ for $x > a$.

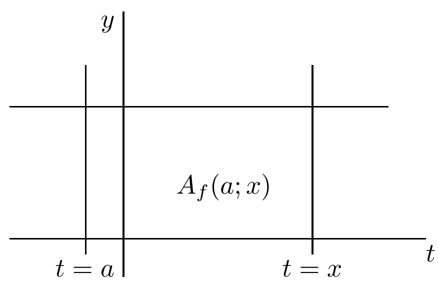

Figure 5.2.11: $A_f(a;x)$ for a positive constant function

Solution: In this case the region under consideration is a rectangle of dimensions $x - a$ and c, since $c > 0$ and $x > a$. We then have that

$$A_f(a;x) = c(x - a), \qquad \text{for all } x \geqslant a. \tag{5.16}$$

We notice that the graph of $A_f(a;x)$ as a function of x is a line that goes through the point in $(a, 0)$ and has slope c as shown in Figure 5.2.12. $\square$

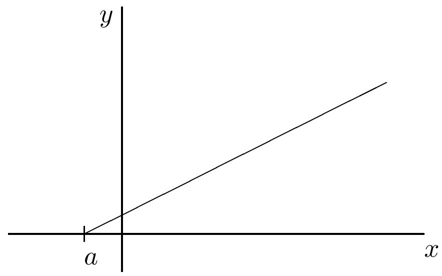

Figure 5.2.12: Sketch of graph of $y = A_f(a; x)$

Definition 5.2.2 (Sign Convention 1)**.** For the case in which f is nonnegative over an interval containing a and x, and x lies to the left of x, we agree to say that $A_f(a; x)$ is the negative of the area of the region that lies below the graph of $y = f(t)$, above the t–axis and between the vertical lines $t = x$ and $t = a$.

Example 5.2.3. Let $f(t) = c$ for all t, where $c > 0$. Compute $A_f(a; x)$ for the case $x < a$.
Solution: Figure 5.2.13 depicts the situation in this case. According to the

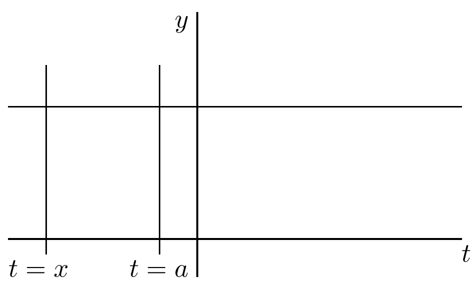

Figure 5.2.13: $A_f(a; x)$ for a positive constant function and $x < a$

sign convention in Definition 5.2.2, $A_f(a; x)$ is the negative of the area of the rectangle with base $[x, a]$ and height c. The area of the rectangle shown in Figure 5.2.13 is

$$(a - x)c,$$

so that

$$A_f(a; x) = -(a - x)c,$$

or

$$A_f(a; x) = c(x - a), \quad \text{for all } x < a. \tag{5.17}$$

□

Note that the formulas in (5.16) and (5.17) are the same. It then follows that, if f is the constant function c, with $c \geqslant 0$,

$$A_f(a;x) = c(x-a), \qquad \text{for } x \in \mathbb{R}. \tag{5.18}$$

The graph of the area function given in (5.18) is shown in Figure 5.2.14, a straight line through the point $(a, 0)$ with slope c.

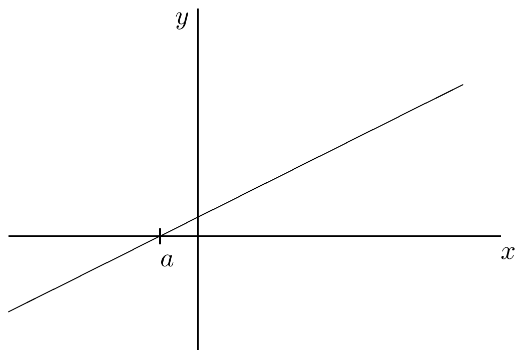

Figure 5.2.14: Sketch of graph of $y = A_f(a;x)$

Next, see what $A_f(a;x)$ is for a piecewise continuous function f whose graph lies below the t–axis. An example would be a constant function $f(t) = c$, for all t, where $c < 0$. A sketch of the graph of this function may be seen in Figure 5.2.15. In order to compute the area function for a function that can be negative

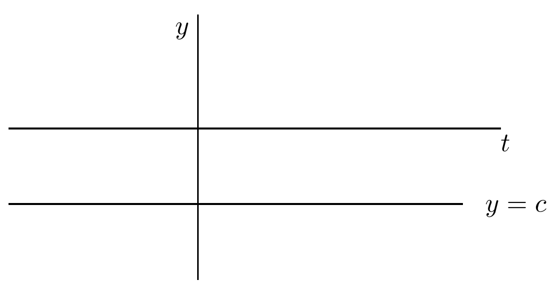

Figure 5.2.15: Graph of $y = c$ for $c < 0$

on parts of its domain, we need a second sign convention:

Definition 5.2.4 (Sign Convention 2)**.** For the case in which f is negative over an interval containing a and x, and x lies to the right of x, we agree to say that $A_f(a;x)$ is the negative of the area of the region that lies below the t–axis, above graph of $y = f(t)$, and between the vertical lines $t = x$ and $t = a$. If x is to the left of a, $A_f(a;x)$ is simply the value of the area of that region.

Example 5.2.5. Let $f(t) = c$ for all t, where $c < 0$. Compute $A_f(a;x)$ for all values of x.

Solution: First, consider the case $x \geqslant a$. Referring to the sketch in Figure 5.2.16, note that the region R in the figure lies below the t–axis. Thus, according

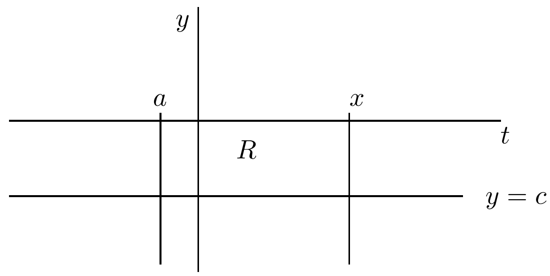

Figure 5.2.16: $A_f(a;x)$ for $x \geqslant a$

to the Sign Convention 2 (see Definition 5.2.4),

$$A_f(a;x) = -\text{area}(R), \tag{5.19}$$

where R is a rectangle of dimensions $x - a$ and $0 - c$; so that

$$\text{area}(R) = -c(x - a). \tag{5.20}$$

Combining (5.19) and (5.20) we see that

$$A_f(a;x) = c(x - a), \qquad \text{for } x \geqslant a. \tag{5.21}$$

For the case $x < a$, refer to Figure 5.2.17. In this case we can apply Sign Conventions 1 and 2 to obtain that

$$A_f(a;x) = -(-\text{area}(R)) = \text{area}(R), \tag{5.22}$$

where R is a rectangle of dimensions $a - x$ and $0 - c$; so that

$$\text{area}(R) = -c(a - x) = c(x - a). \tag{5.23}$$

Combining (5.22) and (5.23) we see that

$$A_f(a;x) = c(x - a), \qquad \text{for all } x < a. \tag{5.24}$$

Thus, in view of (5.21) and (5.24), we conclude that

$$A_f(a;x) = c(x - a), \qquad \text{for } x \in \mathbb{R}, \tag{5.25}$$

which is the same formula in (5.18) obtained for the case $c \geqslant 0$.

The graph of the area function in (5.25) is a straight line through $(a, 0)$ and negative slope, c (see Figure 5.2.18). □

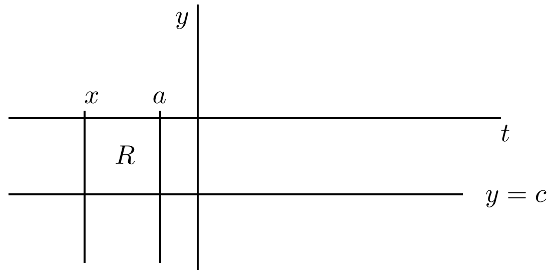

Figure 5.2.17: $A_f(a;x)$ for $x < a$

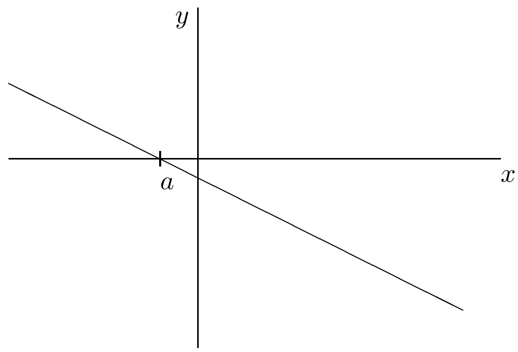

Figure 5.2.18: Sketch of graph of $y = c(x - a)$ for $c < 0$

In the previous examples we have derived the following fact about the area function for a constant function:

Theorem 5.2.6 (Area Function Fact 1)**.** Let $f(t) = c$ for all t, where c is any real number. Then, for any real number a,

$$A_f(a;x) = c(x - a), \quad \text{for } x \in \mathbb{R}. \tag{5.26}$$

Note that the formula in (5.26) can also be written as

$$A_c(a;x) = cx - ca \quad \text{for } x \in \mathbb{R}. \tag{5.27}$$

Example 5.2.7. Let $f(t) = \begin{cases} -1, & \text{for } t < 0; \\ 1, & \text{for } t \geqslant 0. \end{cases}$
Compute $A_f(0;x)$ for all values of x.

Solution: Figure 5.2.19 shows a sketch of the graph of f. The figure also shows

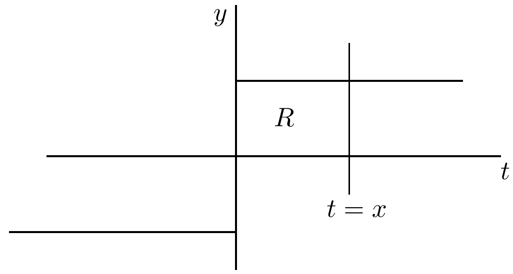

Figure 5.2.19: Sketch of the graph of $y = f(t)$

the line $t = x$ for $x > 0$, and the region R below the graph of f above the t–axis, and between the lines $t = 0$ and $t = x$. In this case, $A_f(0; x)$ is just the area of R; so that,

$$A_f(0; x) = \text{area}(R),$$

where R is a rectangle with dimensions x and 1. Thus,

$$A_f(0; x) = x, \quad \text{for } x \geqslant 0. \tag{5.28}$$

For the case in which $x < 0$, the region R lies below the t–axis, as shown in Figure 5.2.20. Also, the dimensions of the rectangle R are $0 - x$ and $0 - (-1)$,

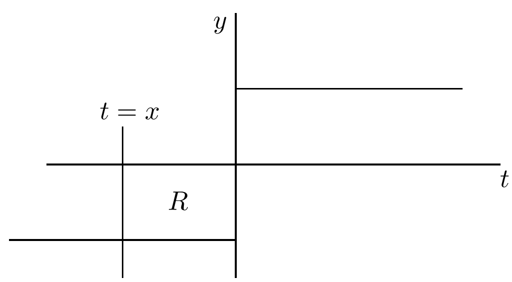

Figure 5.2.20: Sketch of the graph of $y = f(t)$

so that

$$\text{area}(R) = -x.$$

Thus, using the sign conventions, we get that

$$A_f(0; x) = -(-\text{area}(R)) = \text{area}(R) = -x, \quad \text{for } x < 0. \tag{5.29}$$

Combining (5.28) and (5.29) we get that the area function for f from 0 to x is

$$A_f(0;x) = \begin{cases} -x, & \text{for } x < 0; \\ x, & \text{for } x \geqslant 0, \end{cases}$$

or

$$A_f(0;x) = |x|, \quad \text{for all } x \in \mathbb{R}. \tag{5.30}$$

The graph the area function in (5.30) is shown in Figure 5.2.21. □

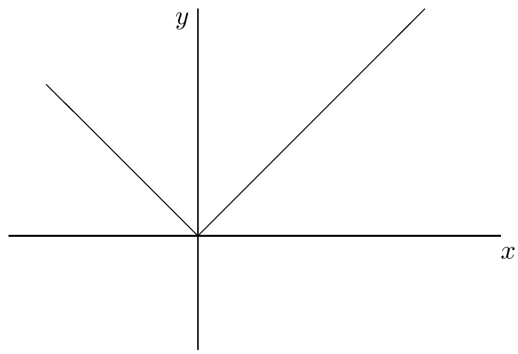

Figure 5.2.21: Sketch of graph of $y = |x|$

Example 5.2.8. Let $f(t) = t$ for all $t \in \mathbb{R}$. Compute $A_f(a;x)$ for any a and any x in $\mathbb{R}$.
Solution: We begin with the case $a > 0$. For $x \geqslant a$, $A_f(a;x)$ is the area of the trapezoidal region R shown in Figure 5.2.22. The area of R can be obtained by subtracting the area of the triangle with vertices $(0,0)$, $(a,0)$ and P from that of the triangle with vertices $(0,0)$, $(x,0)$ and Q. Note the P has Cartesian coordinates (a,a) and Q has Cartesian coordinates (x,x)). We then get

$$A_f(a;x) = \frac{1}{2}x^2 - \frac{1}{2}a^2, \quad \text{for } x \geqslant a. \tag{5.31}$$

For the case $0 \leqslant x < a$, refer to the sketch in Figure 5.2.23. According to the Sign Convention 1, $A_f(A;x)$ is the negative of the area of the trapezoidal region R in Figure 5.2.23. The area of the region R in the figure can be obtained by subtracting the area of the triangle with vertices $(0,0)$, $(x,0)$ and Q from that of the triangle with vertices $(0,0)$, $(a,0)$ and P. Note the P has Cartesian coordinates (a,a) and Q has Cartesian coordinates (x,x)). We then get

$$\text{area}(R) = \frac{1}{2}a^2 - \frac{1}{2}x^2. \tag{5.32}$$

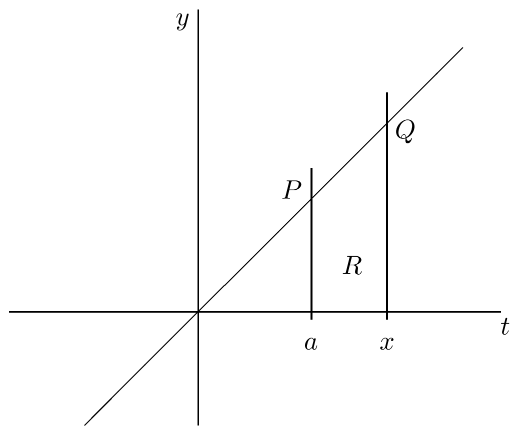

Figure 5.2.22: Sketch of graph of $y = t$

It then follows from (5.32) that

$$A_f(a;x) = -\text{area}(R) = \frac{1}{2}x^2 - \frac{1}{2}a^2, \quad \text{for } 0 \leqslant x < a, \tag{5.33}$$

which is the same formula obtained in (5.2.25) for the case $x \geqslant a$.

Next, consider the case $x < 0$. According the Sign Conventions 1 and 2,

$$A_f(a;x) = -\text{area}(R_1) + \text{area}(R_2), \tag{5.34}$$

where R_1 and R_2 denote the two triangular regions shown in Figure 5.2.24. Given that, in Figure 5.2.24, P has Cartesian coordinates (a, a) and Q has Cartesian coordinates (x, x), it follows that

$$\text{area}(R_1) = \frac{1}{2}a^2 \tag{5.35}$$

and

$$\text{area}(R_2) = \frac{1}{2}(-x)(-x) = \frac{1}{2}x^2. \tag{5.36}$$

Combining (5.34), (5.35) and (5.36) we obtain

$$A_f(a;x) = \frac{1}{2}x^2 - \frac{1}{2}a^2, \quad \text{for } x < 0, \tag{5.37}$$

which is the same formula obtained in (5.31) and (5.33) for the other two cases. We therefore conclude, in view of (5.31), (5.33) and (5.37), that

$$A_f(a;x) = \frac{1}{2}x^2 - \frac{1}{2}a^2, \quad \text{for } x \in \mathbb{R}, \tag{5.38}$$

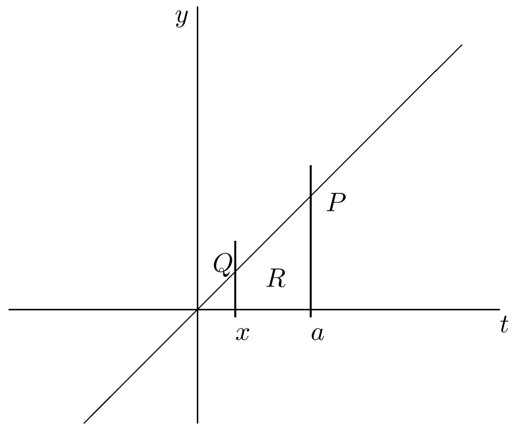

Figure 5.2.23: Calculation of $A_f(A;x)$ for $0 \leqslant x < a$

in the case $a \geqslant 0$.

For the case $a < 0$, note that

$$A_f(a;0) = -A_f(0,-a), \tag{5.39}$$

(see Figure 5.2.25). The triangular regions in Figure 5.2.25 have the same area. However, by the Sign Convention 2, $A_f(a;0)$ is $-\text{area}(R_2)$, which implies (5.39), since $A_f(0;-a)$ is $\text{area}(R_1)$.

Now, we can write

$$A_f(a;x) = A_f(a;0) + A_f(0,x), \quad \text{for any } x \in \mathbb{R} \tag{5.40}$$

(see Problem 1 in Assignment 8). It follows from (5.38) that

$$A_f(0,x) = \frac{1}{2}x^2, \quad \text{for any } x \in \mathbb{R}. \tag{5.41}$$

From (5.39) we obtain that

$$A_f(a,0) = -\frac{1}{2}a^2. \tag{5.42}$$

Finally, combining (5.40), (5.41) and (5.42), we get

$$A_f(a;x) = \frac{1}{2}x^2 - \frac{1}{2}a^2, \quad \text{for } x \in \mathbb{R}, \tag{5.43}$$

in the case $a < 0$ as well. □

We shall state the result in Example 5.2.10 as our second Area Function fact:

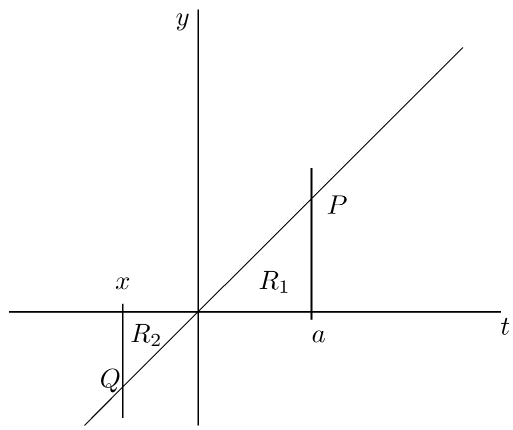

Figure 5.2.24: Calculation of $A_f(A;x)$ for $x<0$

Theorem 5.2.9 (Area Function Fact 2)**.** Let $f(t)=t$ for all $t\in\mathbb{R}$. Then, for any real number a,

$$A_f(a;x)=\frac{1}{2}x^2-\frac{1}{2}a^2,\qquad \text{for } x\in\mathbb{R}. \tag{5.44}$$

We will also write the formula in (5.44) as

$$A_t(a;x)=\frac{1}{2}x^2-\frac{1}{2}a^2,\qquad \text{for } x\in\mathbb{R} \text{ and all } a\in\mathbb{R}. \tag{5.45}$$

In the previous examples we have been able to compute $A_f(a;x)$ by expressing the region under consideration in terms of triangles or rectangles, computing the areas of the components by well known geometric facts, and then adding up the areas of the components. This procedure only works when the boundary of the region is a polygon. If any portion of the boundary of the region is curved (that is, not a straight line segment), the procedure used in Examples 5.2.1 through 5.2.1 no longer works. For example, suppose that $f(t)=t^2$ for all $t\in\mathbb{R}$ and we wish to compute $A_f(a;x)$, where a is some positive number and $x>a$ (see Figure 5.2.26). In this case, $A_f(a;x)=\text{area}(R)$, where R is the region shown in Figure 5.2.26 that lies above the t–axis and below the portion of the parabola with equation $y=t^2$ between the vertical lines $t=a$ and $t=x$. Since the top portion of the boundary of R is not a polygonal curve, we cannot use the decomposition into triangles or rectangles to compute the area of R. We can, however, approximate the area of R by the area of a region bounded by a polygonal curve that can be computed by means of elementary formulas from plane geometry. One example of such an approximating polygonal region is shown in Figure 5.2.27.

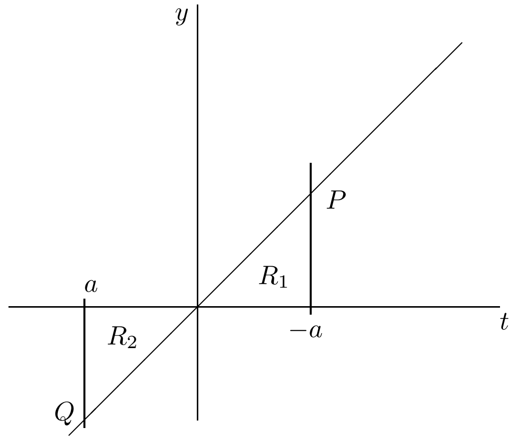

Figure 5.2.25: $A_f(a;0) = -A_f(0,-a)$ for $a < 0$

Example 5.2.10. Let $f(t) = t^2$ for all $t \in \mathbb{R}$. Compute $A_f(a;x)$ for any a and any x in $\mathbb{R}$.
Solution: To solve the problem posed in this example, we use the ideas implemented in the calculation of the area of a circular region of radius r discussed in Example 5.1.3. This time, however, we use a circumscribed polygon, P_n, pictured in Figure 5.2.27. The polygonal region in Figure 5.2.27 is made up of n congruent rectangles obtained as follows: Define a subdivision of the interval $[a,x]$,

$$[t_o,t_1],[t_1,t_2],\ldots,[t_{n-2},t_{n-1}],[t_{n-1},t_n], \quad \text{with } t_o = a \text{ and } t_n = x, \tag{5.46}$$

by

$$t_o = a \quad \text{and} \quad t_k = a + kh, \quad \text{for } k = 1,2,3,\ldots,n, \tag{5.47}$$

where

$$h = \frac{x-a}{n} \tag{5.48}$$

is the length of the base of each sub–rectangle.

The height of the k^{th} rectangle making up P_n, namely the rectangle with base $[t_{k-1},t_k]$, is taken to be $f(t_k)$, the value of the function f at the right–hand end–point of the interval $[t_{k-1},t_k]$ (see Figure 5.2.27). It then follows that

$$\text{area}(P_n) = \sum_{k=1}^{n} f(t_k)h, \quad \text{for each } n \geqslant 2. \tag{5.49}$$

Using the definition of t_k in (5.47) and the formula $f(t) = t^2$, for all $t \in \mathbb{R}$, we

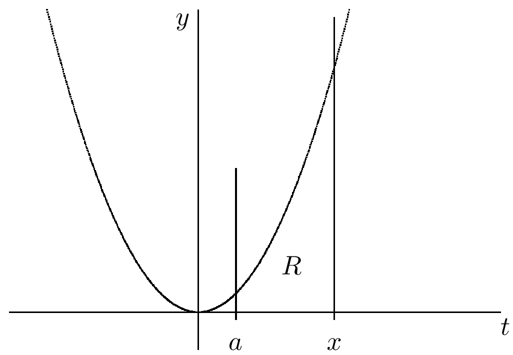

Figure 5.2.26: Sketch of graph of $y = t^2$

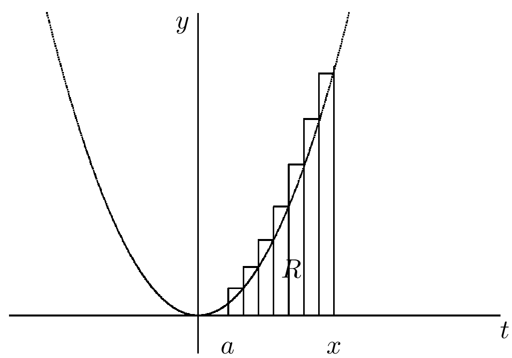

Figure 5.2.27: Polygonal approximation to area(R)

get that

$$\begin{aligned} f(t_k)h &= h(a+kh)^2 \\ &= h[a^2 + 2ahk + h^2k^2] \\ &= ha^2 + 2ah^2k + h^3k^2, \end{aligned} \tag{5.50}$$

for $k = 1, 2, 3, \ldots, n$.

Substituting the results of the calculations in (5.50) into (5.49) yields

$$\text{area}(P_n) = \sum_{k=1}^{n} [ha^2 + 2ah^2k + h^3k^2], \quad \text{for each } n \geqslant 2. \tag{5.51}$$

Using the associative and distributive properties for real numbers arithmetic, we can re–write (5.51) as

$$\text{area}(P_n) = nha^2 + 2ah^2\sum_{k=1}^{n} k + h^3\sum_{k=1}^{n} k^2, \quad \text{for } n \geqslant 1. \tag{5.52}$$

The sums on the right–hand side of (5.52) can be evaluated as follows:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{5.53}$$

(see Problem 4 in Assignment #1), and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{5.54}$$

(see Problem 5 in Assignment #1).

Substituting the sums in (5.53) and (5.54) into the right–hand side of (5.52), and using the definition of h in (5.48), we obtain from (5.52) that

$$\text{area}(P_n) = (x-a)a^2 + a(x-a)^2\frac{n+1}{n} + \frac{1}{6}(x-a)^3\frac{(n+1)(2n+1)}{n^2}, \tag{5.55}$$

for $n \geqslant 1$.

Next, we see what happens as the number of rectangles making up P_n increases; that is as n increases. In view of (5.48), we note that as the number of rectangles increase, the length of their bases decreases; in fact, it follows from (5.48) that

$$\lim_{n\to\infty} h = \lim_{n\to\infty}\frac{x-a}{n} = 0.$$

In order to see what the sequence of approximating areas, $(\text{area}(P_n))$, is doing as $n \to \infty$, we need to evaluate the limits

$$\lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1, \tag{5.56}$$

and

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 1\cdot 2 = 2. \tag{5.57}$$

The fact that the limits in (5.56) and (5.57) exist implies that the sequence of approximations, $(\text{area}(P_n))$, to $\text{area}(R)$ converges to a limit that can be computed by taking the limit as n tends to infinity on both sides of (5.55) and using the results of the calculations in (5.56) and (5.57):

$$\lim_{n\to\infty}\text{area}(P_n) = (x-a)a^2 + a(x-a)^2 + \frac{1}{3}(x-a)^3. \tag{5.58}$$

The expression on the right–hand side of (5.58) can be simplified by means of the binomial expansions

$$(x-a)^2 = x^2 - 2ax + a^2 \tag{5.59}$$

and

$$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3. \tag{5.60}$$

Substituting the expansions in (5.59) and (5.60) into the right–hand side of (5.58) and simplifying then yields

$$\lim_{n\to\infty} \text{area}(P_n) = \frac{1}{3}x^3 - \frac{1}{3}a^3. \tag{5.61}$$

The fact that the limit in (5.61) exists and is given by the expression on the right–hand side of (5.61) suggests that

$$A_f(x;a) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } x \geqslant a, \tag{5.62}$$

and $a > 0$. This statement will be made more precise by the use of the Riemann integral in the next section.

For the case in which $0 \leqslant x < a$, the calculations leading to (5.62) can be used to get that

$$A_f(x;a) = \frac{1}{3}a^3 - \frac{1}{3}x^3, \quad \text{for } 0 \leqslant x < a. \tag{5.63}$$

(Notice that the order of a and x has been switched). Observe that, by virtue of the Sign Convention 1, we can write

$$A_f(a;x) = -A_f(x;a), \quad \text{for } x < a. \tag{5.64}$$

Combining (5.63) and (5.64) then yields

$$A_f(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } 0 \leqslant x < a, \tag{5.65}$$

which is the same formula for $A_f(a;x)$ in (5.62) for the case $x \geqslant a$.

Next, consider the case $x < 0$. Referring to the sketch in Figure 5.2.28, we see that

$$A_f(a;x) = A_f(a;0) + A_f(0;x) \tag{5.66}$$

(see also Problem 1(b) in Assignment #8). Using the Sign Convention 1 we have that

$$A_f(a;0) = -A_f(0;a). \tag{5.67}$$

Also, by the symmetry of the graph of $y = t^2$ with respect to the y-axis and the Sign Convention 1, we can write

$$A_f(0;x) = -A_f(0;-x). \tag{5.68}$$

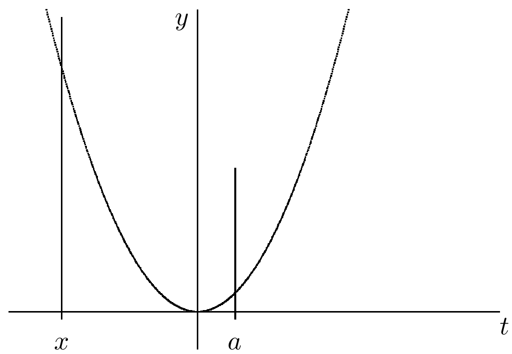

Figure 5.2.28: Calculation of $A_f(a;x)$ for $x<0$

Then, since $a>0$ and $-x>0$, we can use the formula in (5.62) to get from (5.67) and (5.68) that

$$A_f(a;0) = -\frac{1}{3}a^3, \tag{5.69}$$

and

$$A_f(0;x) = -\frac{1}{3}(-x)^3 = \frac{1}{3}x^3. \tag{5.70}$$

since $(-x)^3 = -x^3$.

Combining (5.66), (5.66) and (5.66), we get that

$$A_f(a;x) = -\frac{1}{3}a^3 + \frac{1}{3}x^3 = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } x<0, \tag{5.71}$$

which is the same formula as that in (5.62) and (5.65) for the other two cases.

In view of (5.62), (5.65) and (5.71) we can write that

$$A_f(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for all } x\in\mathbb{R}, \text{ and } a \geqslant 0. \tag{5.72}$$

It remains to see that the formula in (5.72) also works for $a<0$.

If $a<0$, use the symmetry of the graph of $y=t^2$ with respect to the y–axis and the Sign Convention 1 to write

$$A_f(0;a) = -A_f(0;-a). \tag{5.73}$$

(see also (5.67)). Thus, since $-a>0$, we can use the formula in (5.72) to obtain from (5.73) that

$$A_f(0;a) = -\frac{1}{3}(-a)^3 = \frac{1}{3}a^3. \tag{5.74}$$

Use the result in Problem 1(b) in Assignment #8 to write

$$A_f(a;x) = A_f(a;0) + A_f(0;x). \tag{5.75}$$

By the Sign Convention 1 we can write

$$A_f(a;0) = -A_f(0;a) = -\frac{1}{3}a^3, \tag{5.76}$$

where we have also used the result in (5.74). Next, use the result in (5.72) to get that

$$A_f(0;x) = \frac{1}{3}x^3. \tag{5.77}$$

Substitute the results in (5.76) and (5.77) into the right–hand side of (5.75) to conclude that

$$A_f(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for all } x \in \mathbb{R}, \text{ and } a < 0. \tag{5.78}$$

Putting together the results in (5.72) and (5.78), we get that

$$A_f(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } x \in \mathbb{R}, \text{ and } a \in \mathbb{R}. \tag{5.79}$$

□

As we did after the previous two examples, we will list the result of Example 5.2.10 in (5.79) as an Area Function Fact for future reference.

Theorem 5.2.11 (Area Function Fact 3)**.** Let $f(t) = t^2$ for all $t \in \mathbb{R}$. Then, for any real number a,

$$A_f(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } x \in \mathbb{R}. \tag{5.80}$$

We will also write the formula in (5.80) as

$$A_{t^2}(a;x) = \frac{1}{3}x^3 - \frac{1}{3}a^3, \quad \text{for } x \in \mathbb{R} \text{ and all } a \in \mathbb{R}. \tag{5.81}$$

5.3 The Area Function as a Riemann Integral

The procedure outlined in Examples 5.1.3 and 5.2.10 is rather general. In this section we apply it to the case in which we wish to compute $A_f(a;x)$, where f is any piece–wise continuous function on an interval I that contains a. There were two notions that were used in the solutions of the problems in Examples 5.1.3 and 5.2.10. First, there was the notion of approximating the area of the region, R, under consideration by the area of polygonal regions, P_n, which could be calculated by elementary means. Secondly, there was the limiting process

$$\lim_{n\to\infty} \text{area}(P_n),$$

which took place as the areas of the n components making up P_n tended to 0 as n tended to infinity.

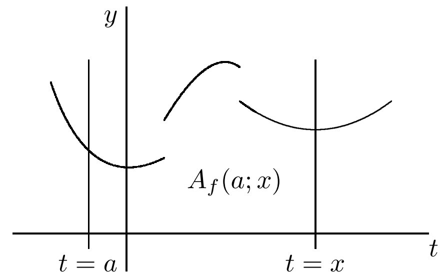

Figure 5.3.29: Graph of piecewise continuous fucntion

For the case of a the region under a piecewise continuous function f, as that pictured in Figure 5.3.29, it is convenient to approximate the signed are $A_f(a;x)$ by the signed area of approximated polygons made up of n rectangles whose bases are the subdivisions

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \quad \text{with } t_o = a \text{ and } t_n = x, \tag{5.82}$$

of the interval $[a, x]$, assuming that $a < x$. In general, the heights of the component rectangles are $f(\tau_k)$, for $k = 1, 2, 3, \ldots, n$, where τ_k is any point in the subinterval $[t_{k-1}, t_k]$. Thus, the ares of the approximating polygons are given by

$$\sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}), \quad \text{for } n = 1, 2, 3, \ldots \tag{5.83}$$

The expressions in (5.83) are known as **Riemann sums** of the function f over the interval $[a, x]$. If we choose the subdivisions in (5.82) in such a way that the largest of the lengths of the subintervals tends to 0 as n tends to infinity, it is reasonable to assume that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}), \tag{5.84}$$

provided that it exists, will yield the value of $A_f(a;x)$. The existence of the limiting value in (5.84) for a piecewise continuous function on the closed and bounded interval $[a, x]$ is a very important fact from analysis. The text for this course refers to it as the *Fundamental Theorem of the Integral Calculus.* In order to avoid confusion with the Fundamental Theorem of Calculus, which will be discussed later in these notes, we shall simply refer to it as the *Existence of the Area Function Theorem.*

Theorem 5.3.1 (Existence of the Area Function)**.** Let f denote a piecewise continuous function defined on an interval containing a point a. Then, for any

x in that interval, $A_f(a:x)$ exists. Furthermore, for the case $x > a$,

$$A_f(a;x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}), \tag{5.85}$$

where

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \qquad \text{with } t_o = a \text{ and } t_n = x, \tag{5.86}$$

is any subdivision of the interval $[a, x]$ with the property that the largest length of the intervals in (5.86) tends to 0 as n tends to infinity, and τ_k is any point in the subinterval $[t_{k-1}, t_k]$.

Definition 5.3.2 (Riemann Integral Notation)**.** The limit on the right–hand side of (5.85), when it exists, is called the Riemann integral of f over the interval $[a, x]$, and it is usually denoted by the symbols $\int_a^x f(t)\ dt$. We then have that

$$A_f(a;x) = \int_a^x f(t)\ dt. \tag{5.87}$$

Example 5.3.3. Using the notation in (5.87) introduced in Definition 5.3.2, the results in the Area Function Facts 1, 2 and 3 can be written as follows

$$\int_a^x c\ dt = cx - ca, \qquad \text{where } c \text{ is constant;} \tag{5.88}$$

$$\int_a^x t\ dt = \frac{1}{2}x^2 - \frac{1}{2}a^2; \qquad \text{and} \tag{5.89}$$

$$\int_a^x t^2\ dt = \frac{1}{3}x^3 - \frac{1}{2}a^3, \tag{5.90}$$

for any a and x in $\mathbb{R}$.

Observe that the equations in (5.88), (5.89) and (5.88) are of the form

$$\int_a^x f(t)\ dt = F(x) - F(a), \tag{5.91}$$

for some function, F. In (5.88) we have

$$F(x) = cx, \qquad \text{for } x \in \mathbb{R},$$

in in the other two cases,

$$F(x) = \frac{1}{2}x^2, \qquad \text{for } x \in \mathbb{R}$$

and

$$F(x) = \frac{1}{3}x^3, \qquad \text{for } x \in \mathbb{R},$$

respectively (see (5.89) and (5.90), respectively). A function F satisfying (5.91) is said to be a **primitive integral** of f, or simply a **primitive** of f.

Definition 5.3.4 (Primitive Integral)**.** Given a piecewise continuous function, f, defined on some interval I, a primitive integral of f is a function, F, satisfying

$$\int_a^x f(t)\ dt = F(x) - F(a), \quad \text{for all } a, x \in I. \tag{5.92}$$

Example 5.3.5. Let f be a piecewise continuous function defined on an interval I; then, given any c in I, the area function, $A_f(c; x)$, is a primitive integral of f. This follows from the identity

$$\int_a^x f(t)\ dt = A_f(c; x) - A_f(c; a);$$

see Problem 2 in Assignment #8.

Suppose that F_1 and F_2 are two primitive integrals of f over the interval I. Then, by (5.92) in Definition 5.3.4,

$$\int_a^x f(t)\ dt = F_1(x) - F_1(a), \quad \text{for all } a, x \in I, \tag{5.93}$$

and

$$\int_a^x f(t)\ dt = F_2(x) - F_2(a), \quad \text{for all } a, x \in I. \tag{5.94}$$

Comparing (5.93) and (5.94) we see that

$$F_2(x) = F_1(x) + F_2(a) - F_1(a), \text{ for all } x \in I. \tag{5.95}$$

Setting $C = F_2(a) - F_1(a)$ we obtain from (5.95) that

$$F_2(x) = F_1(x) + C, \text{ for all } x \in I.$$

Thus, any two primitives of f differ by a constant.

Definition 5.3.6 (Indefinite Integral)**.** Let f be a piecewise continuous function defined on some interval I, and F be a primitive integral of f. The indefinite integral of f, denoted by $\inf f(x)\ dx$, is defined by

$$\int f(x)\ dx = F(x) + C,$$

where C denotes an arbitrary constant.

Example 5.3.7. We have already derived the following indefinite integrals:

$$\begin{aligned}
\int k\ dx &= kx + C, \quad \text{where } k \text{ is a constant;} \\
\int x\ dx &= \frac{1}{2}x^2 + C; \\
\int x^2\ dx &= \frac{1}{3}x^3 + C.
\end{aligned}$$

The last two indefinite integrals in Example 5.3.7 suggest that

$$\int x^m \, dx = \frac{1}{m+1} x^{m+1} + C, \qquad \text{for } m = 1, 2, 3, \ldots \tag{5.96}$$

The integration formula in (5.96) will be derived in Appendix D.

Definition 5.3.8 (Definite Integral)**.** Let f be a piecewise continuous function defined on some interval I containing a and b. The Riemann integral $\displaystyle\int_a^b f(t)\, dt$ is also called the **definite integral** of f from a to b.

If the indefinite integral, or an primitive, of a piecewise continuous function, f, on an interval I is known, we can evaluate the definite integral $\displaystyle\int_a^b f(t)\, dt$, for any a and b in I as follows

$$\int_a^b f(t)\, dt = F(b) - F(a), \tag{5.97}$$

where F is any primitive integral of f in I. The expression in (5.97) is usually written

$$\int_a^b f(t)\, dt = \Big[F(x)\Big]_a^b, \tag{5.98}$$

where the meaning of the right–hand side in (5.98) is given by the right–hand side in (5.97),

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example 5.3.9. Compute the area under the graph of $y = t^2$, above the t–axis, and between the vertical lines $t = -1$ and $t = 2$.
Solution: The region R in question is shown in Figure 5.3.30. The area of R is given by $\displaystyle\int_{-1}^2 t^2\, dt$, so that

$$\text{area}(R) = \left[\frac{1}{3}x^3\right]_{-1}^2 = \frac{1}{3}2^3 - \frac{1}{3}(-1)^3 = \frac{8}{3} + \frac{1}{3} = 3,$$

where we have used the integration formula in (5.96) with $m = 2$. □

Theorem 5.3.10 (Some Properties of the Rieamnn Integral)**.** Let f and g denote piecewise continuous functions defined on an interval containing the points a and b. Then,

(i) $\displaystyle\int_a^b cf(t)\, dt = c\int_a^b f(t)\, dt$, for any constant c.

(ii) $\displaystyle\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt.$

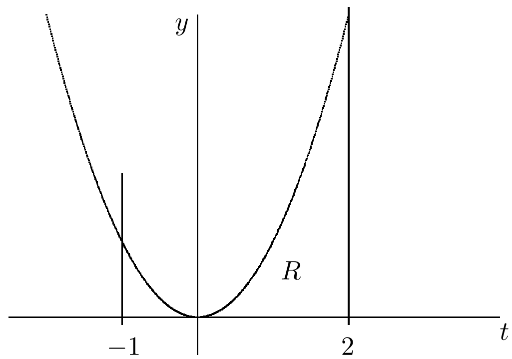

Figure 5.3.30: Region R in Example 5.3.9

The properties in Theorem 5.3.10 can be derived as a consequence of the definition of the Riemann integral and the limit facts for sequences discussed in Section 3.1. In the next two examples we present applications of those properties.

Example 5.3.11. Evaluate the definite integral $\displaystyle\int_{-2}^{1}(8-2t^2)\ dt$.

Solution: Using property (ii) in Theorem 5.3.10, we get

$$\int_{-2}^{1}(8-2t^2)\ dt = \int_{-2}^{1} 8\ dt + \int_{-2}^{1}(-2)t^2\ dt. \tag{5.99}$$

The first integral on the right–hand side of (**??**) can be evaluated using the integration formula in (5.88) to get

$$\int_{-2}^{1} 8\ dt = 8(1-(-2)) = 24. \tag{5.100}$$

In order to evaluate the second integral on the right–hand side of (5.99), first use property (ii) in in Theorem 5.3.10 to get

$$\int_{-2}^{1}(-2)t^2\ dt = -2\int_{-2}^{1} t^2\ dt. \tag{5.101}$$

Next, use the integration formula in (5.96) with $m=2$ to deduce that

$$F(x) = \frac{1}{3}x^3$$

is a primitive integral on $f(t) = t^2$ over $\mathbb{R}$ to get from (5.101) that

$$\begin{aligned} \int_{-2}^{1} (-2)t^2 \, dt &= -2\left[\frac{1}{3}x^3\right]_{-2}^{1} \\ &= -2\left(\frac{1}{3} - \frac{1}{3}(-2)^3\right) \\ &= -\frac{2}{3}(1+8) \\ &= -6. \end{aligned} \tag{5.102}$$

Finally, using the results of the calculations in (5.100) and (5.102), we obtain from (5.99) that

$$\int_{-2}^{1} (8 - 2t^2) \, dt = 24 - 6 = 18.$$

□

Theorem 5.3.12 (More Properties of the Rieamnn Integral)**.** Let f denote a piecewise continuous functions defined on an interval I. Then,

(i) $\displaystyle\int_b^a f(t)\, dt = -\int_a^b f(t)\, dt,$ for any points a and b in I.

(ii) $\displaystyle\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b g(t)\, dt,$ for any points a, b and c in I.

The properties in Theorem 5.3.12 can be derived as a consequence of the area function interpretation of the Riemann integral $\displaystyle\int_a^x f(t)\, dt,$ for any a and x in I (see Problem 1 in Assignment #9).

Example 5.3.13. Evaluate the definite integral $\displaystyle\int_1^3 |t-2|\, dt.$

***Solution**:* Note that

$$|t-2| = \begin{cases} 2-t, & \text{if } t < 2; \\ t-2, & \text{if } t \geqslant 2. \end{cases} \tag{5.103}$$

Thus, we can take advantage of property (ii) in Theorem 5.3.12 to express the integral $\displaystyle\int_1^3 |t-2|\, dt$ as a sum of an integral over the interval $(1,2)$ and an integral over the interval $(2,3)$. We then have, in view of (5.103) that

$$\int_1^3 |t-2|\, dt = \int_1^2 (2-t)\, dt + \int_2^3 (t-2)\, dt. \tag{5.104}$$

Next, we evaluate each of the integrals on the right–hand side of (5.104) separately.

Using property (ii) in Theorem 5.3.10 and the integration formulas in (5.88) and (5.89), we compute

$$\int_1^2 (2-t)\ dt = \left[2x - \frac{1}{2}x^2\right]_1^2 = \left(2(2) - \frac{1}{2}(2)^2\right) - \left(2(1) - \frac{1}{2}(1)^2\right),$$

from which we get that

$$\int_1^2 (2-t)\ dt = (4-2) - \left(2 - \frac{1}{2}\right) = \frac{1}{2}. \tag{5.105}$$

Similarly,

$$\int_2^3 (t-2)\ dt = \left[\frac{1}{2}x^2 - 2x\right]_2^3 = \left(\frac{1}{2}3^2 - 2(3)\right) - \left(\frac{1}{2}2^2 - 2(2)\right),$$

from which we get that

$$\int_2^3 (2-t)\ dt = \left(\frac{9}{2} - 6\right) - (2-4) = \frac{9}{2} - 4 = \frac{1}{2}. \tag{5.106}$$

Substituting the results of the calculations in (5.105) and (5.106) into (5.104) yields

$$\int_1^3 |t-2|\ dt = 1.$$

□

Example 5.3.14. In this example we compute the area function $A_f(a;x)$ for $f(t) = \cos t$, for all $t \in \mathbb{R}$, where $a \in \mathbb{R}$ and $x > a$.

Since cos is continuous on $\mathbb{R}$, the Existence of the Area Function Theorem (Theorem 5.3.1) implies that

$$A_f(a;x) = \lim_{n\to\infty} \sum_{k=1}^{n} \cos(\tau_k)(t_k - t_{k-1}), \tag{5.107}$$

where

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \qquad \text{with } t_o = a \text{ and } t_n = x, \tag{5.108}$$

is any subdivision of the interval $[a, x]$ with the property that the largest length of the intervals in (5.108) tends to 0 as n tends to infinity, and τ_k is any point in the subinterval $[t_{k-1}, t_k]$.

We choose the following subdivision of $[a, x]$:

Set

$$h = \frac{x-a}{n}, \tag{5.109}$$

where n is chosen so that $h < 2\pi$.

Put

$$t_o = a \quad \text{and} \quad t_k = a + kh, \quad \text{for } k = 1, 2, 3, \ldots, n, \tag{5.110}$$

so that

$$t_n = b.$$

Observe that the length of each subinterval is

$$t_k - t_{k-1} = h,$$

so that

$$\lim_{n\to\infty} (t_k - t_{k-1}) = 0,$$

by virtue of (5.109).

Finally, set $\tau_k = t_k = a + kh$, for $k = 1, 2, 3, \ldots, n$. We then have that the conditions of the Existence of the Area Function are fulfilled and so

$$\int_a^x \cos t \; dt = \lim_{n\to\infty} \sum_{k=1}^{n} \cos(a + kh) \cdot h. \tag{5.111}$$

In order to evaluate the limit on the right–hand side of (5.111), we first evaluate the Riemann sums

$$\sum_{k=1}^{n} \cos(a + kh) \cdot h = h \sum_{k=1}^{n} \cos(a + kh). \tag{5.112}$$

Multiply and divide the sum on the right–hand side of (5.112) by $2 \sin\left(\frac{h}{2}\right)$ to get

$$\sum_{k=1}^{n} \cos(a + kh) \cdot h = \frac{h}{2\sin(h/2)} \sum_{k=1}^{n} 2\cos(a + kh) \sin\left(\frac{h}{2}\right). \tag{5.113}$$

Note that we can write

$$\frac{h}{2\sin(h/2)} = \frac{h/2}{\sin(h/2)}; \tag{5.114}$$

and setting

$$\theta = \frac{h}{2}, \tag{5.115}$$

we see from (5.114) that

$$\frac{h}{2\sin(h/2)} = \frac{\theta}{\sin\theta}, \tag{5.116}$$

by virtue of (5.115) and (5.109),

$$\lim_{n\to\infty} \theta = 0. \tag{5.117}$$

It then follows from (5.117) and (5.116) that

$$\lim_{n\to\infty} \frac{h}{2\sin(h/2)} = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1, \tag{5.118}$$

where we have used the limit fact in (3.37) established in Example 3.2.14.

In view of (5.118), (5.111), (5.112) and (5.113), in order to compute the limit in (5.111), it remains to compute the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\cos(a+kh)\sin\left(\frac{h}{2}\right). \tag{5.119}$$

Next, use the trigonometric identity

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta),$$

with $\alpha = a + kh$ and $\beta = h/2$, to re–write the sum in (5.119) as

$$\sum_{k=1}^{n} 2\cos(a+kh)\sin\left(\frac{h}{2}\right) = \sum_{k=1}^{n}\left[\sin\left(a+\left(k+\frac{1}{2}\right)h\right) - \sin\left(a+\left(k-\frac{1}{2}\right)h\right)\right],$$

which simplifies to

$$\sum_{k=1}^{n} 2\cos(a+kh)\sin\left(\frac{h}{2}\right) = \sin\left(x+\frac{3h}{2}\right) - \sin\left(a+\frac{h}{2}\right) \tag{5.120}$$

by virtue of (5.109).

Using (5.109) again, we see that

$$\lim_{n\to\infty} h = 0;$$

hence, it follows from (5.120) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\cos(a+kh)\sin\left(\frac{h}{2}\right) = \lim_{h\to 0}\left[\sin\left(x+\frac{3h}{2}\right) - \sin\left(a+\frac{h}{2}\right)\right],$$

which yields

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\cos(a+kh)\sin\left(\frac{h}{2}\right) = \sin x - \sin a, \tag{5.121}$$

by the continuity of the sine function.

Finally, combining (5.118) and (5.121), we get from (5.113) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \cos(a+kh)\cdot h = \sin x - \sin a. \tag{5.122}$$

Comparing (5.111) and (5.122) we see that

$$\int_a^x \cos t\; dt = \sin x - \sin a. \tag{5.123}$$

We conclude from the result of Example in (5.123) that $F(x) = \sin x$, for all $x \in \mathbb{R}$, is a primitive integral of $f(t) = \cos t$, for $t \in \mathbb{R}$. We also get the integration formula

$$\int \cos x \; dt = \sin x + C. \tag{5.124}$$

Example 5.3.15. Let $f(t) = \cos t$, for all $t \in \mathbb{R}$. Compute the area function $A_f(0; x)$ and sketch the graph of $y = A_f(0; x)$
Solution: Figure 5.3.31 shows a sketch of the graph of $y = \cos t$.

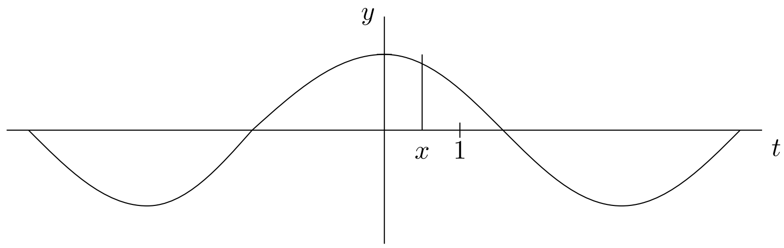

Figure 5.3.31: Sketch of graph of $y = \cos t$

Using the integration formula in (5.123), with $a = 0$, we obtain that

$$A_f(0; x) = \sin x, \quad \text{for } x \in \$,$$

since $\sin(0) = 0$.
A sketch of the graph of $y = A_f(0; x)$ is shown in Figure 5.3.32. □

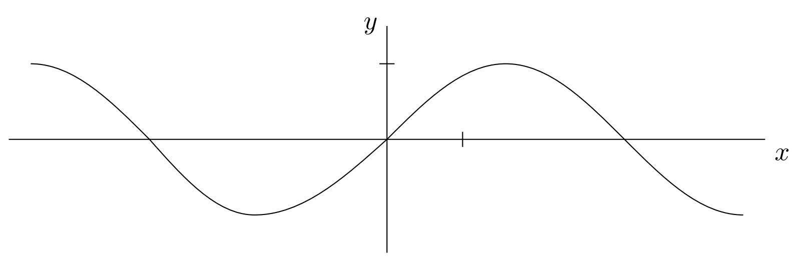

Figure 5.3.32: Sketch of graph of $y = \sin x$

In Examples 5.2.10 and 5.3.14 we chose a special subdivision of the interval $[a, x]$ in which all the subintervals had the same length,

$$h = \frac{x - a}{n},$$

tending to 0 as n tends to infinity. In the next example we will see that, in some cases, it is more convenient to choose the subintervals to have different lengths.

Example 5.3.16. Let $f(t) = \dfrac{1}{t^2}$, for $t > 0$. In this example we compute the area function $A_f(1; x)$, for $x > 0$.

First, we consider the case $x > 1$. Figure 5.3.33 shows a sketch of the graph of f. The figure also shows the region, R, below the graph of f, above the t–axes and between the vertical lines $t = 1$ and $t = x$.

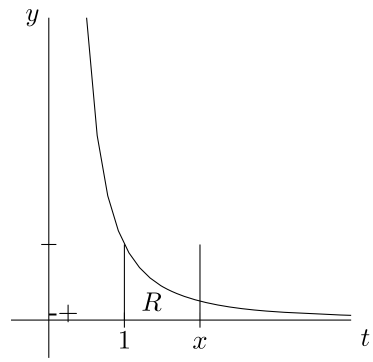

Figure 5.3.33: Sketch of graph of $y = 1/t^2$

We choose a subdivision of $[a, x]$ as follows:

Set

$$q_n = x^{1/n}, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.125}$$

and note that, since x is positive,

$$\lim_{n\to\infty} q_n = 1. \tag{5.126}$$

(The limit fact in (5.126) is proved in Appendix B.1).

Put

$$t_k = q_n^k, \quad \text{for } k = 1, 2, 3, \ldots, n, \tag{5.127}$$

so that

$$t_o = 1 \quad \text{and} \quad t_n = x. \tag{5.128}$$

The length of each subinterval, $[t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$, is given by

$$t_k - t_{k-1} = q_n^k - q_n^{k-1} = q_n^{k-1}(q_n - 1), \quad \text{for } k = 1, 2, 3. \ldots, n. \tag{5.129}$$

It follows from (5.129) and the fact that $q_n > 1$ that

$$t_k - t_{k-1} \leqslant q_n^{n-1}(q_n - 1) = x\frac{q_n - 1}{q_n}, \quad \text{for } k = 1, 2, 3. \ldots, n. \tag{5.130}$$

It follows from (5.126) that

$$\lim_{n\to\infty} x\frac{q_n - 1}{q_n} = 0;$$

thus, in view of (5.130), we see that the conditions of the Existence of the Area Function Theorem are fulfilled.

Next, set $\tau_k = t_k = q_n^k$, for $k = 1, 2, 3, \ldots, n$. Thus, by Theorem 5.3.1,

$$\int_1^x \frac{1}{t^2}\,dt = \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{(q_n^k)^2}\cdot q_n^{k-1}(q_n - 1). \tag{5.131}$$

In order to evaluate the limit on the right–hand side of (5.131), we first evaluate the Riemann sums

$$\sum_{k=1}^{n} \frac{1}{(q_n^k)^2}\cdot q_n^{k-1}(q_n - 1) = \frac{q_n - 1}{q_n}\sum_{k=1}^{n}\frac{1}{q_n^k}, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5.132}$$

The sum on the right-hand side of (5.132) is a geometric sum of the form $\displaystyle\sum_{k=1}^{n} r^k$, whose sum, for $r \neq 1$, is given by

$$\sum_{k=1}^{n} r^k = \frac{r - r^{n+1}}{1 - r}. \tag{5.133}$$

(See Problem 1 in Assignment #11 for a derivation of the formula in (5.133)).

Applying the formula in (5.133) for the case $r = \dfrac{1}{q_n}$, we see that we can rewrite the sum on the right-hand side of (5.132) as

$$\sum_{k=1}^{n}\frac{1}{q_n^k} = \frac{\dfrac{1}{q_n} - \dfrac{1}{q_n^{n+1}}}{1 - \dfrac{1}{q_n}},$$

which simplifies to

$$\sum_{k=1}^{n}\frac{1}{q_n^k} = \frac{1 - \dfrac{1}{x}}{q_n - 1}, \tag{5.134}$$

where we have used the fact $q_n^n = x$, which follows from the definition of q_n in (5.125).

Next, combine (5.132) and (5.134) to get that

$$\sum_{k=1}^{n} \frac{1}{(q_n^k)^2}\cdot q_n^{k-1}(q_n - 1) = \frac{1}{q_n}\left(1 - \frac{1}{x}\right), \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5.135}$$

Using the limit fact in (5.126), we get from (5.135) that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q_n^k)^2}\cdot q_n^{k-1}(q_n-1)=1-\frac{1}{x},$$

from which we get, in view of (5.131), that

$$\int_1^x \frac{1}{t^2}\,dt=-\frac{1}{x}+1,\qquad \text{for } x>1. \tag{5.136}$$

Next, consider the case $0<x<1$. In this case, we proceed as in the case $x>1$; however, this time, we first write

$$A_f(1;x)=\int_1^x\frac{1}{t^2}\,dt=-\int_x^1\frac{1}{t^2}\,dt, \tag{5.137}$$

an then define

$$q_n=\frac{1}{x^{1/n}},\qquad \text{for } n=1,2,3,\ldots, \tag{5.138}$$

and

$$t_k=xq_n^k,\qquad \text{for } k=0,1,2,\ldots,n, \tag{5.139}$$

so that

$$t_0=x\quad\text{ and }\quad t_n=1.$$

We also have that

$$\lim_{n\to\infty}q_n=1. \tag{5.140}$$

The intervals

$$[t_o,t_1],[t_1,t_2],\ldots,[t_{n-2},t_{n-1}],[t_{n-1},t_n] \tag{5.141}$$

form a subdivision of $[x,1]$ with lengths

$$t_k-t_{k-1}=xq_n^{k-1}(q_n-1),\qquad \text{for } k=1,2,3,\ldots,n. \tag{5.142}$$

Since $q_n>1$ for all n, it follows from (5.142) that

$$t_k-t_{k-1}\leqslant xq_n^{n-1}(q_n-1)=\frac{q_n-1}{q_n},\qquad \text{for } k=1,2,3,\ldots,n. \tag{5.143}$$

It follows from (5.140) that

$$\lim_{n\to\infty}\frac{q_n-1}{q_n}=0;$$

thus, in view of (5.143), we see that the conditions of the Existence of the Area Function Theorem are fulfilled in this case as well.

Choosing $\tau_k=t_k=xq_n^k$, for $k=1,2,3,\ldots,n$ this time, we can apply Theorem 5.3.1 again to obtain

$$\int_x^1\frac{1}{t^2}\,dt=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(xq_n^k)^2}\cdot xq_n^{k-1}(q_n-1). \tag{5.144}$$

The sum in the right–hand side of (5.144) can be written as

$$\sum_{k=1}^{n} \frac{1}{(xq_n^k)^2} \cdot xq_n^{k-1}(q_n - 1) = \frac{1}{x}\frac{q_n - 1}{q_n}\sum_{k=1}^{n}\frac{1}{q_n^k}. \tag{5.145}$$

Using the geometric sum formula in (5.133) for the case $r = \dfrac{1}{q_n}$, again, we can rewrite the sum on the right-hand side of (5.145) as

$$\sum_{k=1}^{n}\frac{1}{q_n^k} = \frac{\dfrac{1}{q_n} - \dfrac{1}{q_n^{n+1}}}{1 - \dfrac{1}{q_n}},$$

which simplifies to

$$\sum_{k=1}^{n}\frac{1}{q_n^k} = \frac{1 - x}{q_n - 1}, \tag{5.146}$$

where we have used the fact $q_n^n = \dfrac{1}{x}$, by virtue of (5.139).

Substituting the expression on the right–hand side of (5.146) for the sum in the right–hand side of (5.145) yields

$$\sum_{k=1}^{n} \frac{1}{(xq_n^k)^2} \cdot xq_n^{k-1}(q_n - 1) = \frac{1}{q_n}\left(\frac{1}{x} - 1\right). \tag{5.147}$$

Next, take the limit as n tends to ∞ on both sides of (5.147), while using (5.140), to obtain from (5.144) that

$$\int_x^1 \frac{1}{t^2}\ dt = \frac{1}{x} - 1. \tag{5.148}$$

Finally, combine(5.137) and (5.148)

$$\int_1^x \frac{1}{t^2}\ dt = -\frac{1}{x} + 1, \qquad \text{for } 0 < x < 1, \tag{5.149}$$

which is the same formula obtained for the case $x > 1$ in (5.136).

Noting that $\displaystyle\int_1^1 \frac{1}{t^2}\ dt = 0$, we observe that the formula in (5.136) and (5.149) works for all $x > 0$. We then have

$$\int_1^x \frac{1}{t^2}\ dt = -\frac{1}{x} + 1, \qquad \text{for } x > 0. \tag{5.150}$$

The formula in (5.150) derived in Example 5.3.16 implies that the function

$$F(x) = -\frac{1}{x}, \qquad \text{for } x > 0,$$

is a primitive integral of the function $f(t) = \dfrac{1}{t^2}$, for $t > 0$. We also get the integration formula

$$\int \frac{1}{x^2}\ dx = -\frac{1}{x} + C. \tag{5.151}$$

Writing the formula in (5.151) as

$$\int x^{-2}\ dx = -x^{-1} + C,$$

we see that the integration formula in (5.151) is a special case of the general formula

$$\int x^m\ dx = \frac{1}{m+1}x^{m+1} + C, \quad \text{for } m \neq -1. \tag{5.152}$$

The integration formula in (5.152) will be derived in Appendix D. In the following example we explore what happens in the case $m = -1$.

Example 5.3.17. Let $f(t) = \dfrac{1}{t}$, for $t > 0$. In this example we compute the area function $A_f(1;x)$, for $x > 0$.

We will follow the outline of the solution to the problem in Example 5.3.16. First, we consider the case $x > 1$. Figure 5.3.34 shows a sketch of the graph of f. The figure also shows the region, R, below the graph of f, above the t–axes and between the vertical lines $t = 1$ and $t = x$.

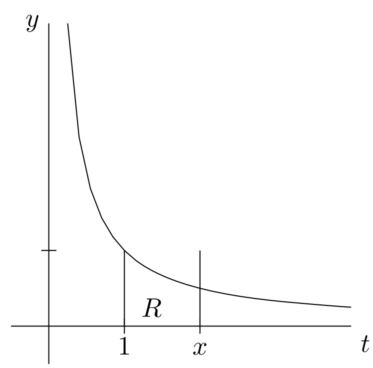

Figure 5.3.34: Sketch of graph of $y = 1/t$

According to the Existence of the Area Function Theorem (Theorem 5.3.1), since $f(t) = \dfrac{1}{t}$ is continuous for $t > 0$, the area of the region R in Figure 5.3.34 is given by

$$\int_1^x \frac{1}{t}\ dt = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\tau_k}(t_k - t_{k-1}), \tag{5.153}$$

where

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \quad \text{with } t_o = 1 \text{ and } t_n = x, \tag{5.154}$$

is any subdivision of the interval $[1, x]$ with the property that the largest length of the intervals in (5.154) tends to 0 as n tends to infinity, and τ_k is any point in the subinterval $[t_{k-1}, t_k]$.

In particular, we can use the subdivision of $[1, x]$ used in Example 5.3.16; namely,

$$t_k = q_n^k, \quad \text{for } k = 0, 1, 2, \ldots, n, \tag{5.155}$$

where

$$q_n = x^{1/n}, \quad \text{for } k = 1, 2, 3, \ldots, \tag{5.156}$$

so that

$$\lim_{n\to\infty} q_n = 1. \tag{5.157}$$

Then, the lengths of the subintervals in (5.154) are given by

$$t_k - t_{k-1} = q_n^{k-1}(q_n - 1), \quad \text{for } k = 1, 2, 3. \ldots, n. \tag{5.158}$$

Now, since $q_n > 1$, it follows from (5.158) that

$$t_k - t_{k-1} \leqslant q_n^{n-1}(q_n - 1), \quad \text{for } k = 1, 2, 3. \ldots, n,$$

or

$$t_k - t_{k-1} \leqslant x\frac{q_n - 1}{q_n}, \quad \text{for } k = 1, 2, 3. \ldots, n, \tag{5.159}$$

where we have used the definition of q_n in (5.157).

Now, it follows from (5.157) that the right–hand side of (5.159) tends to 0 as n tends infinity. Consequently, the conditions for the Existence of the Area Function hold. Thus, taking $\tau_k = t_k = q_n^k$, for $k = 1, 2, 3, \ldots, n$, we get from (5.153) that

$$\int_1^x \frac{1}{t}\, dt = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{q_n^k} q_n^{k-1}(q_n - 1), \tag{5.160}$$

where we have used (5.155) and (5.158).

Next, simplify the sum on the right–hand side of (5.160) to get

$$\int_1^x \frac{1}{t}\, dt = \lim_{n\to\infty} \frac{n(q_n - 1)}{q_n}. \tag{5.161}$$

The limit on the right–hand side of (5.161) is guaranteed to exist by the Existence of the Area Function Theorem. Thus, in view of the limit fact in (5.157), we can write (5.161) as

$$\int_1^x \frac{1}{t}\, dt = \lim_{n\to\infty} [n(x^{1/n} - 1)], \quad \text{for } x > 1. \tag{5.162}$$

where we have used the definition of q_n in (5.157). Calculations for the case $0 < x < 1$ lead to the same formula in (5.162). We then have that

$$\int_1^x \frac{1}{t}\, dt = \lim_{n\to\infty} [n(x^{1/n} - 1)], \qquad \text{for } x > 0. \tag{5.163}$$

The result of Example 5.3.17 in (5.163) shows that the area function, $A_f(1; x)$, for $f(t) = \dfrac{1}{t}$, for $t > 0$, is given by the limit of the sequence $(n(x^{1/n} - 1))$; namely,

$$\lim_{n\to\infty} [n(x^{1/n} - 1)], \qquad \text{for } x > 0. \tag{5.164}$$

We will denote the limit in (5.164) by $\ln(x)$, and call it the **natural logarithm** of x. We then have that

$$\ln(x) = \lim_{n\to\infty} [n(x^{1/n} - 1)], \qquad \text{for } x > 0, \tag{5.165}$$

and, by virtue of (5.163),

$$\ln(x) = \int_1^x \frac{1}{t}\, dt, \qquad \text{for } x > 0. \tag{5.166}$$

We will see later in this section why the function ln defined by limit expression in (5.165), or by the integral in (5.166) is called a logarithm.

We note that the integral formula in (5.166) implies that $\ln(x)$, for $x > 0$, is a primitive integral of $f(t) = \dfrac{1}{t}$, for $t > 0$. We therefore get the integration formula

$$\int \frac{1}{x}\, dx = \ln(x) + C, \qquad \text{for } x > 0. \tag{5.167}$$

Example 5.3.18. Estimate the area of the region below the graph of $y = \dfrac{1}{t}$, for $t > 0$, and above the t–axis from $t = 1$ to $t = 2$.

Solution: Figure 5.3.35 shows a sketch of the region, R, under consideration in this problem. The area of R is given by

$$\text{area}(R) = \int_1^2 \frac{1}{t}\, dt.$$

Thus, according to (5.166),

$$\text{area}(R) = \ln(2). \tag{5.168}$$

In order to estimate the value of $\ln(2)$ in (5.168), we can use (5.165) to write

$$\ln(2) = \lim_{n\to\infty} [n(2^{1/n} - 1)]. \tag{5.169}$$

Table 5.1 shows the values $n(2^{1/n} - 1)$ where n is a power of 10 up to 10^8.

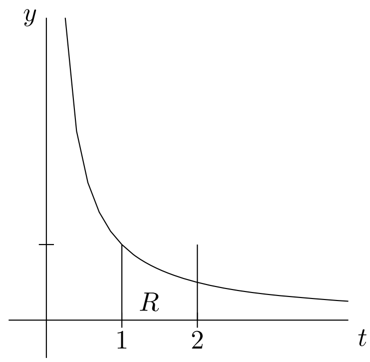

Figure 5.3.35: Sketch of graph of $y = 1/t$

n	$n(2^{1/n} - 1)$
1	1.0000000000
10	0.717734625
100	0.695555006
1000	0.693387463
10^4	0.693171204
10^5	0.693149583
10^6	0.693147421
10^7	0.693147204
10^8	0.693147184

Table 5.1: Values of $n(2^{1/n} - 1)$ for powers of 10

Examination of the last values in the second column of the table shows that an estimate for ln(2), to four decimal places, is 0.6931. In view of (5.168), we then have that

$$\text{area}(R) \doteq 0.6931, \tag{5.170}$$

where the dot on top of the equal sign in (5.170) indicates that the value on the right–hand side is not a exact value for the left–hand side, but an estimate. □

Theorem 5.3.19 (Properties of ln). Let $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$, for $x > 0$.

(i) For any positive numbers a and b,

$$\ln(ab) = \ln(a) + \ln(b) \tag{5.171}$$

(ii) For any positive number, a, and any natural number, n,

$$\ln(a^n) = n \cdot \ln(a). \tag{5.172}$$

To see why (5.171) is true for the case in which $a > 1$ and $b > 1$, observe that

$$1 < a < ab,$$

so that

$$\int_1^{ab} \frac{1}{t}\, dt = \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt,$$

from which we get

$$\ln(ab) = \ln(a) + \int_a^{ab} \frac{1}{t}\, dt. \tag{5.173}$$

Next, we use the Existence of the Area Function Theorem to evaluate the right–most integral in (5.173) as follows:

Set

$$q_n = b^{1/n}, \qquad \text{for } k = 1, 2, 3, \ldots, \tag{5.174}$$

and

$$t_k = aq_n^k, \qquad \text{for } k = 0, 1, 2, \ldots, n. \tag{5.175}$$

We then have that

$$\lim_{n\to\infty} q_n = 1, \tag{5.176}$$

and

$$t_k - t_{k-1} = aq_n^{k-1}(q_n - 1), \qquad \text{for } k = 0, 1, 2, \ldots, n. \tag{5.177}$$

Since $q_n > 1$ for all n, it follows from (5.177) that

$$t_k - t_{k-1} \leqslant aq_n^{n-1}(q_n - 1), \qquad \text{for } k = 0, 1, 2, \ldots, n,$$

from which we get that

$$t_k - t_{k-1} \leqslant b \cdot \frac{q_n - 1}{q_n}, \qquad \text{for } k = 0, 1, 2, \ldots, n, \tag{5.178}$$

by virtue of (5.174).

Note that, as a consequence of (5.176),

$$\lim_{n\to\infty} b \cdot \frac{q_n - 1}{q_n} = 0;$$

Thus, in view of (5.178), the largest of the lengths of the subintervals

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n]$$

tends to 0 as n tends to infinity. Hence, taking $\tau_k = aq_n^k$, for $k = 1, 2, \ldots, n$, we get by applying Theorem 5.3.1 that

$$\int_a^{ab} \frac{1}{t}\, dt = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{aq_n^k} aq_n^{k-1}(q_n - 1). \tag{5.179}$$

Simplifying the sum on the right–hand side of (5.179) we obtain

$$\sum_{k=1}^{n} \frac{1}{aq_n^k} aq_n^{k-1}(q_n - 1) = \sum_{k=1}^{n} \frac{q_n - 1}{q_n} = n\frac{q_n - 1}{q_n}. \tag{5.180}$$

Thus, using (5.174) and (5.176), we obtain from (5.180) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{aq_n^k} aq_n^{k-1}(q_n - 1) = \lim_{n\to\infty} [n(b^{1/n} - 1)]. \tag{5.181}$$

Combining (5.179) and (5.181) we get that

$$\int_a^{ab} \frac{1}{t}\, dt = \lim_{n\to\infty} [n(b^{1/n} - 1)],$$

which, in view of the definition of ln in (5.165), implies that

$$\int_a^{ab} \frac{1}{t}\, dt = \ln(b). \tag{5.182}$$

Combining (5.173) and (5.182) yields (5.171).

In order to establish (5.172), we may proceed by induction on n. First, note that the result is true for $n = 1$ because

$$\ln(a^1) = \ln(a) = 1 \cdot \ln(a).$$

Next, we show that if the result is true for n, then it must also be true for $n+1$. Thus, assume the result is true for n; that is, assume that

$$\ln(a^n) = n \cdot \ln(a), \tag{5.183}$$

and note that $a^{n+1} = a \cdot a^n$. Then, using (5.171),

$$\ln(a^{n+1}) = \ln(a) + \ln(a^n). \tag{5.184}$$

Thus, by virtue of (5.183), we obtain from (5.184) that

$$\ln(a^{n+1}) = \ln(a) + n\ln(a) = (n + 1)\ln(a),$$

which shows that the result in (5.172) is true for $n + 1$.

5.4 Interpretations of the Riemann Integral

We have already seen that the Riemann integral can be used to define the area function for a piecewise continuous function. In this section we present other interpretations of the integral that come up in applications. We begin with the interpretation that we alluded to in the introductory example to these notes: recovering a function from its rate of change.

5.4.1 Recovering a Function from its Rate of Change

In Chapter 2 we alluded to the fact that, if $v(t)$ denotes the speed of an object at time t, and if v is assumed to be a continuous function of t, then the distance traveled by the object over the time interval $[t_o, t]$ is given by

$$s(t) = \int_{t_o}^{t} v(\tau)\ d\tau, \qquad \text{for } t \geqslant t_o. \tag{5.185}$$

We can arrive at (5.185) by first considering a subdivision of the time interval $[t_o, t]$,

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \tag{5.186}$$

where the times t_k are given by

$$t_k = t_o + kh, \qquad \text{for } k = 0, 1, 3, \ldots, n, \tag{5.187}$$

n is a positive integer, and h is the length, $t_k - t_{k-1}$, of each of the subintervals in (5.186); that is,

$$h = \frac{t - t_o}{n}. \tag{5.188}$$

Note that it follows from (5.187) and (5.188) that $t_n = t$.

If n is very large, then, according to (5.188), the interval $[t_{k-1}, t_k]$ is very small, so that, by virtue of the continuity of v, we may approximate the speed, $v(t)$, for any $t \in [t_{k-1}, t_k]$, by a constant value $v(\tau_k)$, where τ_k can be taken to be any time in the interval $[t_{k-1}, t_k]$. We then have that the distance traveled by the object over the interval $[t_{k-1}, t_k]$ can be approximated by $v(\tau_k)(t_k - t_{k-1})$; that is,

$$s(t_k) - s(t_{k-1}) \approx v(\tau_k)(t_k - t_{k-1}). \tag{5.189}$$

The terms on the left–hand side of (5.189) add up to $s(t)$, since $s(t_n) = s(t)$ and $s(t_o) = 0$. Thus, an estimate for $s(t)$, the distance traveled by the vehicle over the interval $[0, t]$, can then be estimated by

$$s(t) \approx \sum_{k=1}^{n} v(\tau_k)(t_k - t_{k-1}), \tag{5.190}$$

By virtue of the continuity of v, the approximations to $s(t)$ on the right–hand side of (5.190) get closer to $s(t)$ as n gets larger and larger. Thus, taking the limit as $n \to \infty$ on the right–hand side of (5.190) and applying Theorem 5.3.1, we obtain

$$s(t) = \int_{t_o}^{t} v(\tau)\ d\tau. \tag{5.191}$$

The integral on the right–hand side of (5.191) is guaranteed to exist by Theorem 5.3.1. Thus, in order to determine the distance traveled by an object moving at a continuous speed, v, it suffice to integrate the function v.

Example 5.4.1. Assume that $v(t) = at$, for all t, where a is a positive constant of proportionality that has units of miles/hr^2. Then, according to the formula in (5.191), the distance traveled by the object over the time interval $[t_o, t]$ is given by

$$s(t) = \int_{t_o}^{t} a\tau \; d\tau,$$

which we can compute using the property (i) in Theorem 5.3.10 and the integration fact in (5.96) to obtain

$$s(t) = a\left[\frac{1}{2}\tau^2\right]_{t_o}^{t} = \frac{a}{2}t^2 - \frac{a}{2}t_o^2, \quad \text{ for } t \geqslant t_o.$$

Example 5.4.2. The speed of an object moving in a straight line is given by the function $v(t) = 25 - 2t$ in meters per second, where the time t is measured in seconds. Compute the distance traveled by the object from time $t = 0$ till it stops.

***Solution**:* The object stops when $v(t) = 0$, or $t = \dfrac{25}{2}$ seconds. The distance traveled by the object over the interval $[0, t]$ is

$$\begin{aligned} s(t) &= \int_0^t v(\tau) \; d\tau \\ &= \int_0^t (25 - 2\tau) \; d\tau \\ &= \left[25\tau - \tau^2\right]_0^t, \end{aligned}$$

so that

$$s(t) = 25t - t^2, \quad \text{ for } t \geqslant 0. \tag{5.192}$$

Substituting $t = \dfrac{25}{2}$ in equation (5.192) yields

$$s\left(\frac{25}{2}\right) = \frac{(25)^2}{2} - \frac{(25)^2}{4} = \frac{(25)^2}{4},$$

so that the object travels 156.25 meters from $t = 0$ till it stops. □

The procedure outlined above to recover the distance traveled by an object from its speed as a function of time can also be applied to a quantity, $Q(t)$, whose rate of change, $R(t)$, is known at all times. Suppose that R is a continuous function of time and that the amount $Q(t)$ is known at some time t_o; that is, $Q(t_o)$ is known. We would like to know what $Q(t)$ is for all $t \geqslant t_o$.

If $t = t_1$ is such that the length of the interval $[t_o, t_1]$ is very small, then we can approximate the rate of change of Q over the interval $[t_o, t_1]$ by a constant rate $R(\tau_1)$, where τ_1 is any time in $[t_o, t_1]$. It follows that the quantity Q increases or decreases approximately by

$$R(\tau_1)(t_1 - t_o).$$

It then follows that the amount Q at time t_1 is $Q(t_o) + R(\tau_1)(t_1 - t_o)$, approximately, or

$$Q(t_1) \approx Q(t_o) + R(\tau_1)(t_1 - t_o),$$

which can be re–written as

$$Q(t_1) - Q(t_o) \approx R(\tau_1)(t_1 - t_o). \tag{5.193}$$

An argument similar to that leading to (5.193) can be used to obtain a sequence of estimates

$$Q(t_k) - Q(t_{k-1}) \approx R(\tau_k)(t_k - t_{k-1}), \quad \text{for } k = 1, 2, 3, \ldots, n, \tag{5.194}$$

where

$$t_o < t_1 < t_2 < t_3 < \cdots t_{n-1} < t_n = t \tag{5.195}$$

is a succession of times with the property that the largest of the time intervals, $t_k - t_{k-1}$, for $k = 1, 2, 3, \ldots, n$, tends to 0 as n tends to infinity, and τ_k is any time in the interval $[t_{k-1}, t_k]$.

Adding the expressions in (5.194) and using (5.195) we obtain the estimate

$$Q(t) - Q(t_o) \approx \sum_{k=1}^{n} R(\tau_k)(t_k - t_{k-1}). \tag{5.196}$$

Letting n tend to infinity on the right—hand side of (5.196) and using Theorem 5.3.1, we see that the right—hand side of (5.196) tends to $\displaystyle\int_{t_o}^{t} R(\tau)\ d\tau$, in view of the assumption that the rate, R, is a continuous function of t. The continuity of R also allows us to conclude that the approximations in (5.196) tend to the exact representation

$$Q(t) - Q(t_o) = \int_{t_o}^{t} R(\tau)\ d\tau,$$

or

$$Q(t) = Q(t_o) + \int_{t_o}^{t} R(\tau)\ d\tau, \quad \text{for } t \geqslant t_o. \tag{5.197}$$

The expression in (5.197) answers in the positive the question of whether we can recover a function from its rate of change in the case in which the rate of change is assumed to be continuous.

Example 5.4.3. Suppose that the rate of change of a quantity, Q, of a substance is proportional t for $t \geqslant 1$. Given that $Q(1) = 2$, compute $Q(t)$ for all $t \geqslant 1$.

Solution: The rate of change of Q with respect to t, denoted by R, is given by

$$R(t) = \frac{K}{t}, \qquad \text{for } t \geqslant 0,$$

for some constant of proportionality K.

Using the formula in (5.197) we have that

$$Q(t) = Q(1) + \int_1^t \frac{K}{\tau}\, d\tau = 2 + K \ln t, \qquad \text{for } t \geqslant 0.$$

□

5.4.2 Computing a Quantity from its Density

Imagine the quantity of a substance is distributed in a cylindrical region with axis along the x–axis over an interval $[a, b]$, as shown in Figure 5.4.36. Assume

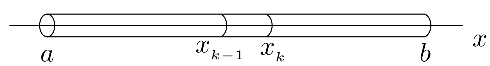

Figure 5.4.36: Cylindrical Rod

the cross sectional area of the cylinder is a constant, A. Assume also that there is continuous function of x, denoted by ρ, with the property that $\rho(x)/A$ gives the density of the substance at x in units of amount of substance per volume; so that $\rho(x)$ is in units of amount of substance per length. The function ρ is called a **linear density**. It then follows that the amount of the substance in a small section of the cylindrical rod over the interval $[x_{k-1}, x_k]$ shown in Figure 5.4.36 is given, approximately, by

$$\frac{\rho(\tau_k)}{A} \cdot A(x_k - x_{k-1}) = \rho(\tau_k) \cdot (x_k - x_{k-1}), \tag{5.198}$$

where τ_k is any point in the interval $[x_{k-1}, x_k]$.

Next, consider a subdivision

$$[x_o, x_1], [x_1, x_2], \ldots, [x_{n-2}, x_{n-1}], [x_{n-1}, x_n], \tag{5.199}$$

where $x_o = a$ and $x_n = b$, of the interval $[a, b]$ with the property that the largest length of the intervals in (5.199) tends to 0 as n tends to infinity. Then, and approximation to the total quantity, Q, of substance in the rod is sum of the approximations in (5.198),

$$Q \approx \sum_{k=1}^{n} \rho(\tau_k) \cdot (x_k - x_{k-1}). \tag{5.200}$$

Letting n tend to infinity on the right–hand side of (5.200), using assumption of continuity of ρ, and applying Theorem 5.3.1, we get from (5.200) that

$$Q = \int_a^b \rho(x)\ dx. \tag{5.201}$$

Example 5.4.4. A rod of length 0.5 meter lies along the x–axis with one end at 0. Suppose the rod is made up of material whose linear density, ρ, varies with x according to the formula

$$\rho(x) = 1 + 2\sqrt{x}, \qquad \text{for } x \geqslant 0,$$

in grams per meter, where x is measured in meters. Then, according to (5.201), the total mass, M, of the rod is given by

$$M = \int_0^{1/2} (1 + 2\sqrt{x})\ dx = \left[x + \frac{2}{3/2}x^{3/2}\right]_0^{1/2} = \frac{1}{2} + \frac{4}{3}\frac{1}{2\sqrt{2}},$$

so that the mass of the rod is

$$M = \frac{1}{2} + \frac{\sqrt{2}}{3} \qquad \text{grams.}$$

5.4.3 Average Value of a Function

Given a piecewise continuous function, f, defined on some interval I, which contains a and b with $a < b$, the **average value** of the function f over $[a, b]$ is defined by

$$\frac{1}{b-a}\int_a^b f(t)\ dt. \tag{5.202}$$

We will denote the value in (5.202) by $\overline{f}$, so that

$$\overline{f} = \frac{1}{b-a}\int_a^b f(t)\ dt, \tag{5.203}$$

with the understanding that $\overline{f}$ also depends on the interval, $[a, b]$, over which the calculations are done.

Example 5.4.5. Compute the average value of $f(t) = 1 - t^2$ over the interval $[-1, 1]$.
Solution: The graph of f over $[-1, 1]$ is shown in Figure 5.4.37. Using the formula in (5.203) we obtain that

$$\overline{f} = \frac{1}{1-(-1)}\int_{-1}^1 (1 - t^2)\ dt = \frac{1}{2}\int_{-1}^1 (1 - t^2)\ dt. \tag{5.204}$$

Thus, using the symmetry of the graph of f with respect to the y–axis, we obtain from (5.204) that

$$\overline{f} = \int_0^1 (1 - t^2)\ dt = \left[t - \frac{1}{3}t^3\right]_0^1 = \frac{2}{3}.$$

□

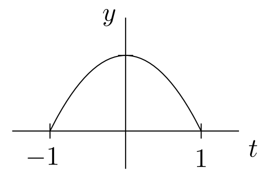

Figure 5.4.37: Sketch of graph of $y = 1 - t^2$

Remark 5.4.6 (Interpretation of the Average Value)**.** For the case in which $f(t) \geqslant 0$ for all $t \in [a, b]$, the average value of f over $[a, b]$ represents the height, $\overline{y}$, of a rectangle over $[a, b]$ with area $\displaystyle\int_a^b f(t)\ dt$; that is,

$$\overline{y}(b - a) = \int_a^b f(t)\ dt,$$

from which we get the formula

$$\overline{y} = \frac{1}{b - a} \int_a^b f(t)\ dt,$$

for the average value of f in (5.203). This is illustrated in Figure 5.4.38 for the function in Example 5.4.5.

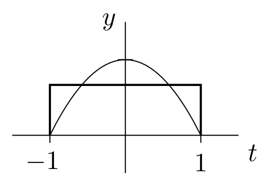

Figure 5.4.38: $\overline{y}(b - a) = \int_a^b f(t)\ dt$

Example 5.4.7 (Average Rate of Change)**.** We saw in Section 5.4.1 that, if R is a continuous function of t that gives the rate of change of a function, f, at any time t in the interval $[a, b]$, then

$$f(b) = f(a) + \int_a^b R(t)\ dt, \tag{5.205}$$

(see the formula in (5.197) applied to $Q = f$, $t_o = a$, $f = b$).

It follows from (5.205) that

$$\frac{1}{b - a} \int_a^b R(t)\ dt = \frac{f(b) - f(a)}{b - a}; \tag{5.206}$$

in other words, the average rate of change of f over $[a, b]$, for $a < b$, is given by $\dfrac{f(b) - f(a)}{b - a}$.

We will denote the left–hand side of (5.205) by $\overline{R}_f(a, b)$, so that

$$\overline{R}_f(a, b) = \frac{1}{b - a} \int_a^b R(t)\ dt, \quad \text{for } a \neq b, \tag{5.207}$$

and, in view of (5.206),

$$\overline{R}_f(a, b) = \frac{f(b) - f(a)}{b - a}, \quad \text{for } a \neq b. \tag{5.208}$$

The right–hand side of (5.208) is called the **difference quotient** of f over $[a, b]$. This will be the starting point in our study of the differential Calculus in the next chapter in these notes.

Chapter 6

Differential Calculus

In Example 5.4.7 of the previous chapter we saw that that the average rate of change for continuous function, f, over and interval $[a, b]$, for $a \neq b$, is given by the equation in (5.208); namely,

$$\overline{R}_f(a, b) = \frac{f(b) - f(a)}{b - a}, \quad \text{for } a \neq b, \tag{6.1}$$

(see also the expression in (5.208)). Thus, the expression in on the right–hand of (6.1), known as the **difference quotient** of f from a to b, gives the average rate of change of f from a to b. In this chapter we will see how to go from an average rate of change to an **instantaneous rate of change**.

6.1 Instantaneous Rate of Change

In this section we solve the inverse problem to the one introduced in Chapter 2 and solved in Section 5.4.1: Given a function f, can we determine its rate of change, R?

We begin with the average rate of change over a small interval $[t, t + h]$, where $h > 0$, or $[t + h, t]$ for $h < 0$. For the case $h > 0$, we have, according to (6.1), that the average rate of change of f from t to $t + h$ is

$$\overline{R}_f(t, t + h) = \frac{f(t + h) - f(t)}{h}. \tag{6.2}$$

We postulate that, if the expression in (6.2) has a limit as $h \to 0$, then the limit will be the rate of change of f at t, or the **instantaneous rate of change** of f at t.

Definition 6.1.1 (Instantaneous Rate of Change)**.** Let f be a function defined on an open interval I and $t \in I$. If the limit

$$\lim_{h \to 0} \overline{R}_f(t, t + h) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \tag{6.3}$$

exists, we call it the **instantaneous rate of change** of f at t. If the limit in (6.3) exists, we denote it by $f'(t)$, and call $f'(t)$ the **derivative** of f at t. We then have that

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}, \tag{6.4}$$

provided that the limit on the right–hand side of (6.4) exists.

Example 6.1.2. Let $f(t) = \sqrt{t}$ for $t \geqslant 0$. Show that the instantaneous rate of change of f exists for all $t > 0$, and compute the derivative $f'(t)$ for all $t > 0$.
Solution: We first compute the difference quotient

$$\begin{aligned} \frac{f(t+h) - f(t)}{h} &= \frac{\sqrt{t+h} - \sqrt{t}}{h} \\ &= \frac{\sqrt{t+h} - \sqrt{t}}{h} \cdot \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \\ &= \frac{(\sqrt{t+h})^2 - (\sqrt{t})^2}{h(\sqrt{t+h} + \sqrt{t})} \\ &= \frac{t+h-t}{h(\sqrt{t+h} + \sqrt{t})}, \end{aligned}$$

for $h \neq 0$, so that

$$\frac{f(t+h) - f(t)}{h} = \frac{1}{\sqrt{t+h} + \sqrt{t}}, \qquad \text{for } h \neq 0. \tag{6.5}$$

Next, use the fact that the function f is continuous and $t > 0$ to see that the limit as $h \to 0$ of the right–hand side of (6.5) exists and equals

$$\lim_{h\to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t+0} + \sqrt{t}} = \frac{1}{2\sqrt{t}}, \qquad \text{for } t > 0. \tag{6.6}$$

Combining (6.5) and (6.6) we see that

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \frac{1}{2\sqrt{t}}, \qquad \text{for } t > 0, \tag{6.7}$$

so that, according to (6.4) in Definition 6.1.1, the instantaneous rate of change of $f(t) = \sqrt{t}$, for $t \geqslant 0$, exists for all $t > 0$, and is given by

$$f'(t) = \frac{1}{2\sqrt{t}}, \qquad \text{for } t > 0.$$

□

Example 6.1.3. Let $f(t) = |t|$ for $t \in \mathbb{R}$. Show that the instantaneous rate of change of f does not exist at $t = 0$.

Solution: In this case the difference quotient at 0 is

$$\frac{f(0+h)-f(0)}{h} \quad = \quad \frac{|h|}{h}, \qquad \text{for } h \neq 0;$$

so that

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} -1, & \text{if } h < 0; \\ +1, & \text{if } h > 0. \end{cases}$$

Thus, $\displaystyle\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1$, while $\displaystyle\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = +1$. It then follows that $\displaystyle\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. Consequently, the rate of change of $f(t) = |t|$, for $t \in \mathbb{R}$, does not exist at $t = 0$. □

6.2 Differentiable Functions

Definition 6.2.1 (Differentiable Functions)**.** A real valued function, f, defined on an open interval, I, is said to be differentiable at $t \in I$ if the limit

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \tag{6.8}$$

exists. If the limit in (6.8) exists at every $t \in I$, then f is said to be differentiable in I.

The concept of a differentiable function is a very important one in Calculus and it applications. We have already seen that the problem of finding the instantaneous rate of a change for a function, f, at time, t, is solved by computing the limit in (6.8) at t, provides that the limit exists. Hence, the problem of computing instantaneous rates of change can be solved for differentiable functions. In a subsequent section in these notes we will give other interpretations of the derivative of a differentiable function. In this section we present several examples of differentiable functions. We also discuss a few properties of differentiable functions.

Example 6.2.2 (Derivative of a Constant Function)**.** Let $f(t) = c$, for all $t \in \mathbb{R}$, where c is a constant. Then, f is differentiable for all $t \in \mathbb{R}$ and $f'(t) = 0$ for all t. This example is worked out in Problem 3 of Assignment #13.

Example 6.2.3 (Derivative of the Identity Function)**.** Let $f(t) = t$, for all $t \in \mathbb{R}$. Then, f is differentiable for all $t \in \mathbb{R}$ and $f'(t) = 1$ for all t. This example is worked out in Problem 4 of Assignment #13.

Example 6.2.4 (Derivative of a Power Function)**.** Let $f(t) = t^n$, for all $t \in \mathbb{R}$. Then, f is differentiable for all $t \in \mathbb{R}$ and $f'(t) = nt^{n-1}$ for all t.
Solution: To see why f is differentiable for all $t \in \mathbb{R}$, first compute the difference quotient

$$\frac{f(t+h)-f(t)}{h} \quad = \quad \frac{(t+h)^n - t^n}{h}, \qquad \text{for } h \neq 0. \tag{6.9}$$

Next, use the factorization fact in Appendix A,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots ab^{n-2} + b^{n-1}),$$

to write

$$(t+h)^n - t^n = (t+h-t)[(t+h)^{n-1} + (t+h)^{n-2}t + (t+h)^{n-3}t^2 + \cdots + (t+h)t^{n-2} + t^{n-1}],$$

from which we get that

$$(t+h)^n - t^n = h[(t+h)^{n-1} + (t+h)^{n-2}t + \cdots + (t+h)t^{n-2} + t^{n-1}]. \quad (6.10)$$

Dividing both sides of the equation in (6.10) by $h \neq 0$ yields

$$\frac{(t+h)^n - t^n}{h} = (t+h)^{n-1} + (t+h)^{n-2}t + \cdots + (t+h)t^{n-2} + t^{n-1}, \quad (6.11)$$

for $h \neq 0$. Note that that, for fixed $t \in \mathbb{R}$, the right–hand side of (6.11) is a polynomial in h. Since polynomials are continuous, it follows that the limit of the expression on the right–hand side of (6.11) as $h \to 0$ exists, and

$$\lim_{h\to 0}[(t+h)^{n-1} + (t+h)^{n-2}t + \cdots + (t+h)t^{n-2} + t^{n-1}] = nt^{n-1}, \quad (6.12)$$

since there are n terms on the right–hand side of (6.11) all tending to t^{n-1} as $h \to 0$.

Combining (6.11) and (6.12) yields

$$\lim_{h\to 0}\frac{(t+h)^n - t^n}{h} = nt^{n-1},$$

which shows that $f(t) = t^n$, for $t \in \mathbb{R}$, is differentiable at every $t \in \mathbb{R}$ and

$$f'(t) = nt^{n-1}, \quad \text{for all } t \in \mathbb{R},$$

which was to be shown. $\square$

Theorem 6.2.5 (Some Properties of Differentiable Functions)**.** Let f and g denote real–valued functions defined in some open interval, I.

(i) Suppose that f is differentiable at $t \in I$. Then, for any constant, c, the function cf is differentiable at t, and

$$(cf)'(t) = cf'(t).$$

(ii) Suppose that f and g are differentiable at t. Then, the function $f+g$ is differentiable at t and

$$(f+g)'(t) = f'(t) + g'(t).$$

Proof of (i): First, compute the difference quotient

$$\frac{(cf)(t+h)-(cf)(t)}{h} = \frac{cf(t+h)-cf(t)}{h}, \quad \text{for } h \neq 0,$$

so that

$$\frac{(cf)(t+h)-(cf)(t)}{h} = c\cdot\frac{f(t+h)-f(t)}{h}, \quad \text{for } h \neq 0, \tag{6.13}$$

where we have factored out c in the numerator in the right–hand side of (6.13).

Note that, since we are assuming that f is differentiable at t, the limit as $h \to 0$ of the difference quotient in the right–hand side of (6.13) exists. It then follows from Function Limit Fact 1 and (ii) of Function Limit Fact 3 on page 22 in these notes that the limit as $h \to 0$ of the difference quotient in the left–hand side of (6.13) exists and

$$\lim_{h\to 0}\frac{(cf)(t+h)-(cf)(t)}{h} = c\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = cf'(t). \tag{6.14}$$

The limit expression in (6.14) shows that cf is differentiable at t and

$$(cf)'(t) = cf'(t),$$

which was to be shown. ■

Proof of (ii): First, compute the difference quotient

$$\begin{aligned}\frac{(f+g)(t+h)-(f+g)(t)}{h} &= \frac{f(t+h)+g(t+h)-(f(t)+g(t))}{h}\\ &= \frac{f(t+h)-f(t)+g(t+h)-g(t)}{h},\end{aligned}$$

for $h \neq 0$, which can be re-written as

$$\frac{(f+g)(t+h)-(f+g)(t)}{h} = \frac{f(t+h)-f(t)}{h}+\frac{g(t+h)-g(t)}{h}, \tag{6.15}$$

for $h \neq 0$.

Note that, since we are assuming that f and g are differentiable at t, the limit as $h \to 0$ of the difference quotients in the right–hand side of (6.15) exist. It then follows from (i) of Function Limit Fact 3 on page 22 in these notes that the limit as $h \to 0$ of the sum of the difference quotients in the right–hand side of (6.15) exists and therefore

$$\lim_{h\to 0}\frac{(f+g)(t+h)-(f+g)(t)}{h} = f'(t)+g'(t), \tag{6.16}$$

by the definition of the derivatives of f and g at t. The limit expression in (6.16) shows that $f+g$ is differentiable at t and

$$(f+g)'(t) = f'(t)+g'(t),$$

which was to be shown. ■

Example 6.2.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote the polynomial function defined by

$$f(t) = 2t^3 - 3t^2 + t - 5, \qquad \text{for } t \in \mathbb{R}. \tag{6.17}$$

Then, by the results in Examples 6.2.2 and 6.2.4 and the Differentiable Functions Facts (i) and (ii) in Theorem 6.2.5, f is differentiable in $\mathbb{R}$, and

$$f'(t) = 2(3)t^2 - 4(2)t + 1 - 0 = 6t^2 - 8t + 1, \qquad \text{for } t \in \mathbb{R}. \tag{6.18}$$

Thus, the derivative of the third–degree polynomial, f, given in (6.17) is the second–degree polynomial, f', given in (6.18).

Remark 6.2.7. The result of Example 6.2.6 holds true for a general n^{th}–degree polynomial, $p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_t t^2 + a_1 t + a_o$, where $a_o, a_1, a_2, \ldots, a_n$ are real constants with $a_n \neq 0$. The polynomial function, p, is differentiable for all $t \in \mathbb{R}$, and its derivative, p', is the polynomial of degree $n-1$ given by $p'(t) = na_n t^{n-1} + (n-1)a_{n-1}t^{n-2} + \cdots + 2a_2 t + a_1$, for all $t \in \mathbb{R}$ (see Problem 2 in Assignment #14).

The difference quotient,

$$\frac{f(t+h) - f(t)}{h}, \qquad \text{for } h \neq 0,$$

is also denoted by the symbol

$$\frac{\Delta f}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}, \qquad \text{for } \Delta t \neq 0, \tag{6.19}$$

read, "the change in f over the change in t." It follows from (6.19) and Definition 6.2.1 that, if f is differentiable at t, then

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = f'(t). \tag{6.20}$$

The limit expression in the left–hand side of (6.20) is often denoted by the symbol $\dfrac{df}{dt}$. Thus,

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}, \tag{6.21}$$

provided that f is differentiable at t.

Definition 6.2.8 (Differential Notation)**.** The symbol df in the left–hand side of (6.21) is called the **differential** of f, and dt the differential of t. The symbol $\dfrac{d}{dt}$ is is an example of a differential operator—it operates on a differentiable function, f, to yield its derivative f'. Thus, in view of (6.21), if f is differentiable at t, then

$$\frac{d}{dt}[f] = \frac{df}{dt} = f'(t).$$

Remark 6.2.9. Using the differential notation in Definition 6.2.8 the properties in Theorem 6.2.5 can be written as

(i) $\dfrac{d}{dt}[cf] = c\dfrac{df}{dt}$, for any constant c, whenever f is differentiable.

(ii) $\dfrac{d}{dt}[f+g] = \dfrac{df}{dt} + \dfrac{dg}{dt}$, whenever f and g are differentiable.

6.3 Interpretations of the Derivative

We have already seen that the derivative of a differentiable function, f, of time, t, gives the instantaneous rate of change of f at t, $f'(t)$. There are other interpretations of the derivative. We begin with the concept of a linear approximation to a differentiable function.

6.3.1 Linear Approximation of a Differentiable Function

Suppose that f is a real valued function of a single variable, t, defined in an open interval, I. Assume that f is differentiable at $a \in I$; then, by virtue of Definition 6.2.1,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a). \tag{6.22}$$

Setting $t = a + h$, we can rewrite (6.22) as

$$\lim_{t\to a} \frac{f(t) - f(a)}{t - a} = f'(a). \tag{6.23}$$

The limit expression in (6.23) is in turn equivalent to

$$\lim_{t\to a} \left| \frac{f(t) - f(a)}{t - a} - f'(a) \right| = 0,$$

which can be written as

$$\lim_{t\to a} \left| \frac{f(t) - f(a) - f'(a)(t-a)}{t - a} \right| = 0,$$

or

$$\lim_{t\to a} \frac{|f(t) - f(a) - f'(a)(t-a)|}{|t - a|} = 0. \tag{6.24}$$

Write

$$E_f(a;t) = f(t) - f(a) - f'(a)(t-a), \quad \text{for } t \in I. \tag{6.25}$$

It follows from (6.25) and (6.24) that, if f is differentiable at a, then

$$\lim_{t\to a} \frac{|E_f(a;t)|}{|t - a|} = 0. \tag{6.26}$$

Solving for $f(t)$ in (6.25) we obtain

$$f(t) = f(a) + f'(a)(t-a) + E_f(a;t), \quad \text{for } t \in I, \tag{6.27}$$

where the term $E_f(a;t)$ satisfies the limit condition in (6.26), provided that f is differentiable at $a \in I$.

The expression in (6.26) implies that, if f is differentiable at a, then the term $E_f(a;t)$ tends to 0 as t tends to a. Thus, for values of t very close to a, we get the approximation

$$f(t) \approx f(a) + f'(a)(t-a), \quad \text{for } t \text{ in } I \text{ very close to } a. \tag{6.28}$$

The expression on the right–hand side of (6.28) is known as the **linear approximation** to f at a. We will denote it by

$$L_f(a;t) = f(a) + f'(t)(t-a), \quad \text{for } t \in \mathbb{R}. \tag{6.29}$$

Using the definition of L_f in (6.29) we obtain from (6.27) that, if f is differentiable at a, then

$$f(t) = L_f(a;t) + E_f(a;t), \quad \text{for } t \in I, \text{ where } \lim_{t\to a} \frac{|E_f(a;t)|}{|t-a|} = 0. \tag{6.30}$$

We can re–write (6.30) by introducing the “little–o” notation.

Definition 6.3.1 (Little–o Notation)**.** A quantity $E(h)$ is said to be little–o of $|h|$, written

$$E(h) = o(|h|),$$

if and only if

$$\lim_{h\to 0} \frac{|E(h)|}{|h|} = 0.$$

Thus, in view of Definition 6.3.1, if f is differentiable at a, then

$$f(t) = L_f(a;t) + o(|t-a|), \quad \text{for } t \in I, \tag{6.31}$$

where

$$L_f(a;t) = f(a) + f'(t)(t-a), \quad \text{for } t \in \mathbb{R},$$

is the linear approximation of f at a. Thus, differentiability a a implies that f can be approximated by a linear function at a in the sense given in (6.31).

Example 6.3.2. Let $f(t) = 4 - t^2$ for all $t \in \mathbb{R}$. We compute the linear approximatively, $L_f(1;t)$, of f at 1,

$$L_f(1;t) = f(1) + f'(1)(t-1), \quad \text{for } t \in \mathbb{R}, \tag{6.32}$$

where

$$f'(t) = \frac{d}{dt}[4-t^2] = \frac{d}{dt}[4] - \frac{d}{dt}[t^2] = -2t, \quad \text{for } t \in \mathbb{R},$$

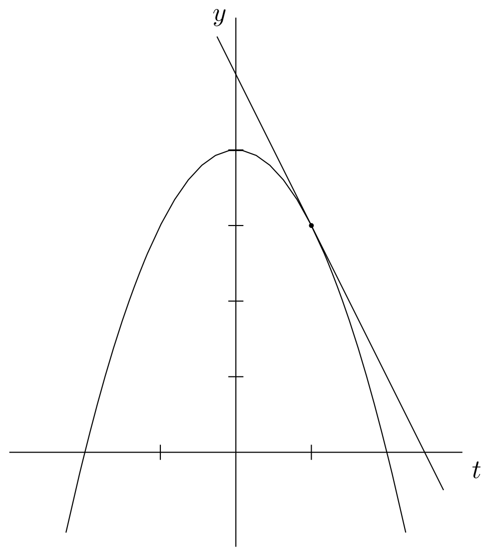

Figure 6.3.1: Sketch of graph of $y = f(t) = 4 - t^2$ and $y = L_f(1;t)$

so that $f'(1) = -2$. We then get from (6.32) that

$$L_f(1;t) = 3 - 2(t-1), \quad \text{for } t \in \mathbb{R},$$

or

$$L_f(1;t) = 5 - 2t, \quad \text{for } t \in \mathbb{R}.$$

Figure 6.3.1 shows a sketch shown the graphs of $y = f(t)$ and its linear approximation at 1.

Figure 6.3.1 in Example 6.3.2 gives a graphical illustration of the meaning of the statement in (6.28) for a function f that is differentiable at a. In the case of the function f in Example 6.3.2 we have, with $a = 1$:

$$f(t) \approx 3 - 2(t-1), \quad \text{for } t \text{ close to } 1,$$

or

$$4 - t^2 \approx 5 - 2t, \quad \text{for } t \text{ close to } 1.$$

Observe that the graphs of $y = 4-t^2$ and $y = 5-2t$ are almost indistinguishable for values of t very close to $t = 1$.

6.3.2 Tangent Line to a Curve

An examination of the graphs in Figure 6.3.1 in Example 6.3.2 suggests that the line $y = L_f(1;t)$ meets the graph of $y = f(t)$ at exactly one point; namely, the point $(1,3)$. We can verify this statement algebraically by solving the equation

$$4 - t^2 = 5 - 2t,$$

or

$$t^2 - 2t + 1 = 0,$$

or

$$(t-1)^2 = 0,$$

which yields exactly one solution: $t = 1$. Thus, the only point of intersection of the curves $y = 4 - t^2$ and $y = 5 - 2t$ is the point $(1,3)$. We say that $y = 5 - 2t$ is **tangent** to the curve $y = 4 - t^2$ at the point $(1,3)$. Observe that the slope of the tangent line, $y = 5 - 2t$, to the graph of $y = f(t)$ at $a = 1$ is $f'(1) = -2$. In other words, the derivative of f at 1 gives the slope of the tangent line to the graph of $y = f(t)$ at $(1, f(1))$. We will make this observation the basis for the definition of the tangent line to the graph of a differentiable function, f, at a point $(a, f(a))$.

Definition 6.3.3 (Tangent Line to the Graph of a Differentiable Function)**.** Let f denote a real valued function defined in an open interval, I, of the real line containing a point a. Assume that f is differentiable at a. Then, the derivative of f at a gives the slope of the tangent line to the graph of $y = f(x)$ in the xy–plane over the interval I. The equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a). \tag{6.33}$$

Example 6.3.4. Give the equation of the tangent line to the graph of $y = \sin x$ at the point $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$.

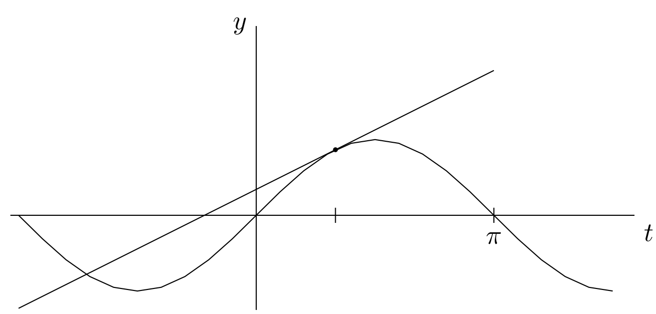

Figure 6.3.2: Sketch of graphs of $y = \sin x$ and its tangent line at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

Solution: Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. We first derive a formula for computing $f'(x) = \sin'(x)$ for all $x \in \mathbb{R}$.

Using the trigonometric identity

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

to compute the difference

$$\begin{aligned} \sin(x+h) - \sin x &= \sin x\cos h + \cos x\sin h - \sin x \\ &= (\cos h - 1)\sin x + \sin h\cos x, \end{aligned}$$

so that, dividing by $h \neq 0$,

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\cos h - 1}{h}\sin x + \frac{\sin h}{h}\cos x, \quad \text{for } h \neq 0. \tag{6.34}$$

Using the result in (3.25) in Example 3.2.12 we have that

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \tag{6.35}$$

Similarly, using the limit fact derived in Example 3.2.14,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1. \tag{6.36}$$

Thus, combining (6.34), (6.35) and (6.34), we conclude that the limit as $h \to 0$ of the difference quotient in the left–hand side of (6.36) exists and

$$\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x, \quad \text{for all } x \in \mathbb{R}, \tag{6.37}$$

where we have used the Function Limit Fact 3 in Theorem 3.2.7 on page 22 in these notes.

We conclude from (6.38) that sin is differentiable in $\mathbb{R}$ and

$$\sin'(x) = \cos x, \quad \text{for all } x \in \mathbb{R}. \tag{6.38}$$

The equation of the tangent to the graph of $y = \sin x$ at the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ is given by (6.33) with $a = \frac{\pi}{3}$ and $f = \sin$:

$$y = \sin\left(\frac{\pi}{3}\right) + \sin'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right),$$

or, using (6.38),

$$y = \frac{\sqrt{3}}{2} + \cos\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right),$$

or

$$y = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right). \tag{6.39}$$

Figure 6.3.2 shows a sketch of the graphs of $y = \sin x$ and its tangent line at the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$. Notice that the tangent line given by (6.33) in Definition 6.3.3 meets the sine cure in the figure at two points and not just at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$. However, when we focus on some small interval around $\frac{\pi}{3}$, the line in (6.39) meet the graph of $y = \sin x$ at exactly one point. □

6.4 Properties of the Derivative

We have already seen that of if f and g are differentiable at t, then so are the functions cf and $f + g$, where c is a constant. Furthermore,

$$\frac{d}{dt}[cf] = c\frac{df}{dt} \tag{6.40}$$

and

$$\frac{d}{dt}[f + g] = \frac{df}{dt} + \frac{df}{dt}. \tag{6.41}$$

Properties (6.40) and (6.41) and very useful in determining the differentiability properties functions obtained by adding multiples of differentiable functions (e.g., polynomial functions).

Example 6.4.1. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = 3t^2 + 2\sin t$, for all $t \in \mathbb{R}$, is differentiable because it is the sum of multiples of differentiable function and

$$f'(t) = \frac{d}{dt}[3t^2 + 2\sin t] = 6t + 2\cos t, \qquad \text{for all } t \in \mathbb{R},$$

It this section we study three additional properties of differentiable functions. We begin by considering products of differentiable functions.

6.4.1 Products of Differentiable Functions

Let f and g denote functions defined on an open interval I. Assume that f and g are differentiable at $t \in I$. Then we can write

$$f(t + h) = f(t) + f'(t)h + E_f(h), \qquad \text{for } |h| \text{ sufficiently small}, \tag{6.42}$$

where

$$E_f(h) = o(|h|). \tag{6.43}$$

Similarly,

$$g(t+h) = g(t) + g'(t)h + E_g(h), \qquad \text{for } |h| \text{ sufficiently small,} \tag{6.44}$$

where

$$E_g(h) = o(|h|). \tag{6.45}$$

Multiplying the expressions in (6.42) and (6.44) yields

$$\begin{aligned} f(t+h)g(t+h) &= (f(t) + f'(t)h + E_f(h))(g(t) + g'(t)h + E_g(h)) \\ &= f(t)g(t) + f(t)g'(t)h + f(t)E_g(h) \\ &\quad + f'(t)g(t)h + f'(t)g'(t)h^2 + f'(t)E_g(h)h \\ &\quad\quad + g(t)E_f(h) + g'(t)E_f(h)h + E_f(h)E_g(h), \end{aligned}$$

from which we get that

$$\begin{aligned} f(t+h)g(t+h) - f(t)g(t) &= h[f(t)g'(t) + f'(t)g(t) + f'(t)E_g(h) + g'(t)E_f(h)] \\ &\quad + f(t)E_g(h) + f'(t)g'(t)h^2 \\ &\quad\quad + g(t)E_f(h) + E_f(h)E_g(h), \end{aligned}$$

which yields

$$\begin{aligned} \frac{f(t+h)g(t+h) - f(t)g(t)}{h} &= f(t)g'(t) + f'(t)g(t) \\ &\quad + f'(t)E_g(h) + g'(t)E_f(h) + f(t)\frac{E_g(h)}{h} \\ &\quad + f'(t)g'(t)h + g(t)\frac{E_f(h)}{h} + E_f(h)\frac{E_g(h)}{h}, \end{aligned}$$

for $h \neq 0$.

Observe that all the terms on the right–hand side of the previous equation, except for the first two, tend to 0 as $h \to 0$, in view of (6.43) and (6.45). Consequently, the limit as $h \to 0$ of the difference quotient $\dfrac{f(t+h)g(t+h) - f(t)g(t)}{h}$ exists and

$$\lim_{h\to 0} \frac{f(t+h)g(t+h) - f(t)g(t)}{h} = f(t)g'(t) + f'(t)g(t);$$

thus, if f and g are differentiable at t, then the product function $fg\colon I \to \mathbb{R}$ given by

$$(fg)(t) = f(t)g(t), \qquad \text{for all } t \in \mathbb{R},$$

is differentiable at t and

$$(fg)'(t) = f(t)g'(t) + f'(t)g(t). \tag{6.46}$$

The property in (6.46) is usually referred to as the **product rule**. We can rewrite it as

$$\frac{d}{dt}[fg] = f\frac{dg}{dt} + \frac{df}{dt}g. \tag{6.47}$$

Example 6.4.2. The function $f\colon \mathbb{R}\to\mathbb{R}$ given by $f(t)=t^2\sin t$, for all $t\in\mathbb{R}$, is differentiable because it is the product of differentiable function and, using (6.47),

$$\begin{aligned} f'(t) &= \frac{d}{dt}[t^2\sin t] \\ &= t^2\frac{d}{dt}[\sin t]+\frac{d}{dt}[t^2]\sin t \\ &= t^2\cos t+2t\sin t, \end{aligned}$$

for all $t\in\mathbb{R}$.

6.4.2 Compositions of Differentiable Functions

Let f g be two functions such that $g(t)$ lies in the domain of f for all t in the domain of g. Then, according to Definition 4.1.8, the composition of f and g, denoted $f\circ g$, is defined and

$$f\circ g(t)=f(g(t)), \quad \text{for } t \text{ in the domain of } g. \tag{6.48}$$

We will see in this section that, if g is differentiable at t, and f is differentiable a $g(t)$, then the composition, $f\circ g$, is differentiable at t.

Suppose that g is differentiable at t so that

$$g(t+h)=g(t)+g'(t)h+E_g(h), \quad \text{for } |h| \text{ sufficiently small,} \tag{6.49}$$

where

$$E_g(h)=o(|h|). \tag{6.50}$$

Similarly, if f is differentiable at $g(t)$, setting $u=g(t)$, we can write

$$f(u+v)=f(u)+f'(u)v+E_f(v), \quad \text{for } |v| \text{ sufficiently small,} \tag{6.51}$$

where

$$E_f(v)=o(|v|). \tag{6.52}$$

Now, using the definition of $f\circ g$ in (6.48), we compute

$$\begin{aligned} f\circ g(t+h) &= f(g(t+h)) \\ &= f(g(t)+g'(t)h+E_g(h)), \end{aligned} \tag{6.53}$$

where we have used (6.49).

Setting

$$u=g(t) \tag{6.54}$$

and

$$v=g'(t)h+E_g(h), \tag{6.55}$$

we can rewrite the result in (6.53) as

$$f \circ g(t+h) \quad = \quad f(u+v). \tag{6.56}$$

Note that, by virtue of (6.50) and (6.55), $|v| \to 0$ as $|h| \to 0$; thus, using (6.51) we can rewrite (6.56) as

$$f \circ g(t+h) \quad = \quad f(u) + f'(u)v + E_f(v), \text{ for } |h| \text{ small.} \tag{6.57}$$

Substituting (6.54) and (6.55) into (6.57), we obtain

$$f \circ g(t+h) = f(g(t)) + f'(g(t))(g'(t)h + E_g(h)) + E_f(v),$$

for $|h|$ small, so that

$$f \circ g(t+h) - f \circ g(t) = f'(g(t))g'(t)h + f'(g(t))E_g(h) + E_f(v), \tag{6.58}$$

for $|h|$ small. Dividing both sides of (6.58) by $h \neq 0$ we get

$$\frac{f \circ g(t+h) - f \circ g(t)}{h} = f'(g(t))g'(t) + f'(g(t))\frac{E_g(h)}{h} + \frac{E_f(v)}{h}, \tag{6.59}$$

for $0 < |h|$ small.

Observe that

$$\lim_{h \to 0} \frac{E_g(h)}{h} = 0, \tag{6.60}$$

by virtue of (6.50). To see what happens to the term $\dfrac{E_f(v)}{h}$, write

$$\frac{E_f(v)}{h} = \frac{E_f(v)}{v} \cdot \frac{v}{h}, \qquad \text{for } h \neq 0,$$

from which we get

$$\frac{|E_f(v)|}{|h|} = \frac{|E_f(v)|}{|v|} \cdot \frac{|v|}{|h|}, \qquad \text{for } h \neq 0, \tag{6.61}$$

after taking absolute values.

Now, it follows from (6.55) and the triangle inequality that

$$|v| \leqslant |g'(t)||h| + |E_g(h)|, \tag{6.62}$$

so that, after division by $|h| > 0$,

$$\frac{|v|}{|h|} \leqslant |g'(t)| + \frac{|E_g(h)|}{|h|}, \qquad \text{for } h \neq 0. \tag{6.63}$$

Combining (6.61) and (6.63) we get

$$\frac{|E_f(v)|}{|h|} \leqslant \frac{|E_f(v)|}{|v|}\left[|g'(t)| + \frac{|E_g(h)|}{|h|}\right], \qquad \text{for } h \neq 0. \tag{6.64}$$

Next, apply the Squeeze Theorem to the inequality in (6.64), together with (6.62), (6.60) and (6.52), to conclude that

$$\lim_{h\to 0}\frac{E_f(v)}{h} = 0. \tag{6.65}$$

it follows from (6.60) and (6.65) that the limit as h tends to zero of the expression on the right–hand side of (6.59) exists and equals $f'(g(t))\cdot g'(t)$. Consequently,

$$\lim_{h\to 0}\frac{f\circ g(t+h) - f\circ g(t)}{h} = f'(g(t))\cdot g'(t),$$

which shows that $f\circ g$ is differentiable at t and

$$(f\circ g)'(t) = f'(g(t))\cdot g'(t),$$

or

$$\frac{d}{dt}[f(g(t))] = f'(g(t))\cdot\frac{d}{dt}[g(t)]. \tag{6.66}$$

The expression in (6.66) is usually referred to as the **Chain Rule**.

Example 6.4.3. The function $f\colon\mathbb{R}\to\mathbb{R}$ given by $f(t) = \sin(t^2)$, for all $t\in\mathbb{R}$, is differentiable because it is the composition of the sine function and the the polynomial function $p(t) = t^2$, for $t\in\mathbb{R}$, both of which are differentiable everywhere. Using (6.66) we obtain

$$\begin{aligned} f'(t) &= \frac{d}{dt}[\sin(t^2)] \\ &= \sin'(t^2)\cdot\frac{d}{dt}[t^2] \\ &= 2t\cos(t^2), \end{aligned}$$

for all $t\in\mathbb{R}$.

Example 6.4.4. Let g be a real valued function defined in some open interval, I. Assume that g is differentiable at $a\in I$ an that $g(a)\neq 0$. We consider the function

$$f(t) = \frac{1}{g(t)},$$

for t near a such that $g(t)\neq 0$.

Chapter 7

Fundamental Theorems

This chapter brings together three major concepts that we have studied in these notes: continuity, the Riemann integral, and the derivative. One thing that these concepts have in common is that they were all defined using the notion of limit; as such, they belong to realm of Calculus. There is also a very important connection between these concepts that was hinted by the observation that integration yields a function from its rate change, provided that the rate of change is continuous.

7.1 Recovering a Function from its Rate

We saw in Section 5.4.1 that, if R is the rate of change of a function, f, of time, t, over some interval I, and R is continuous over I, then, for any $a \in I$,

$$f(t) = f(a) + \int_a^t R(\tau)\ d\tau, \quad \text{for all } t \in I. \tag{7.1}$$

In Section 6.1 we learned that the rate of change of a differentiable function, f, over an open interval, I, is given by its derivative, f'; in other words, it R denotes the rate of change of a function, f, over the interval, I, then

$$R(t) = f'(t) \quad \text{for all } t \in I. \tag{7.2}$$

Combining the results in (7.1) and (7.2), together with the requirement that f' be continuous on I, we obtain the result

$$f(t) = f(a) + \int_a^t f'(\tau)\ d\tau, \quad \text{for all } t \in I,$$

provided that f' is continuous on I. In these notes, we shall refer to this result as the the **First Fundamental Theorem of Calculus** or FTC I, for short.

Theorem 7.1.1 (First Fundamental Theorem of Calculus (FTC I))**.** Let f be a differentiable function defined in an open interval I containing a. Suppose that f' is continuous on I. Then,

$$f(t) = f(a) + \int_a^t f'(\tau)\ d\tau, \quad \text{for all } t \in I.$$

Theorem 7.1.1 provides a complete solution to the the problem that we posed at the start of these notes for the case in which the rate is a continuous function.

As an application of the First Fundamental Theorem of Calculus we present the following example.

Example 7.1.2. Let f denote a differentiable function defined over some interval I. Assume that $f'(t) = 0$ for all $t \in I$. We show that f must be constant in I.

Pick any point $a \in I$ and apply the First Fundamental Theorem of Calculus to get

$$f(t) = f(a) + \int_a^t f'(\tau)\ d\tau = f(a) + 0 = f(a), \quad \text{for all } t \in I,$$

since $f'(t) = 0$ for all $t \in I$; thus, f is constant on I.

7.2 Differentiability of the Area Function

Let f denote a continuous function of an open interval I. For a fixed $a \in I$, we saw in Section 5.3 how to define the area function, $A_f(a;x)$, for $x \in I$, in terms of a Riemann integral:

$$A_f(a;x) = \int_a^x f(t)\ dt, \quad \text{for all } x \in I.$$

The Second Fundamental Theorem of Calculus (FTC II) states that, if f is continuous on I, then the area function if differentiable and its derivative is the function f:

$$\frac{d}{dx}[A_f(a;x)] = f(x), \quad \text{for } x \in I,$$

or

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x), \quad \text{for } x \in I.$$

Theorem 7.2.1 (Second Fundamental Theorem of Calculus (FTC II))**.** Let f be a continuous function defined in an open interval I containing a. Then, the function

$$G(x) = \int_a^x f(t)\ dt, \quad \text{for all } x \in I,$$

is differentiable in I and $G'(x) = f(x)$ for all $x \in I$, or

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x), \quad \text{for } x \in I.$$

Remark 7.2.2. A proof of Theorem 7.2.1 may be found in Appendix D of these notes. However, the idea of the proof is not hard to understand if we observe that the difference quotient

$$\frac{A_f(a; x+h) - A_f(a; x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\, dt, \qquad \text{for } h > 0 \text{ small}, \tag{7.3}$$

is the average value of f over the interval $[x, x+h]$. Denoting this value by $\overline{f}[x, x+h]$, we can rewrite (7.3) as

$$\frac{A_f(a; x+h) - A_f(a; x)}{h} = \overline{f}[x, x+h], \qquad \text{for } h > 0 \text{ small}. \tag{7.4}$$

The continuity of f at x implies that

$$\lim_{h \to 0} \overline{f}[x, x+h] = f(x). \tag{7.5}$$

It follows from (7.5) and (7.4) that

$$\lim_{h \to 0} \frac{A_f(a; x+h) - A_f(a; x)}{h} = f(x),$$

which shows that the area function, $A_f(a; x)$, is differentiable at x and its derivative is $f(x)$.

Example 7.2.3. The function $f(t) = \dfrac{1}{t}$, for $t > 0$, is continuous on the interval $I = (0, \infty)$. It then follows by the Second Fundamental Theorem of Calculus that the natural logarithm function,

$$\ln(x) = \int_1^x \frac{1}{t}\, dt, \qquad \text{for } x > 0,$$

is differentiable on I and

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}, \qquad \text{for } x > 0. \tag{7.6}$$

The formula in (7.6) in Example 7.2.3 provides a very useful differentiation formula which we state below.

$$\frac{d}{du}[\ln(u)] = \frac{1}{u}, \qquad \text{for } u > 0. \tag{7.7}$$

Example 7.2.4. Let $f(x) = \ln(1 + x^2)$, for all $x \in \mathbb{R}$. Then, f is the composition of the natural logarithm function, ln, and the polynomial function, $p(x) = 1 + x^2$, for $x \in \mathbb{R}$, both of which are differentiable in their domains.

Hence, f is differentiable and, applying the Chain Rule, its derivative is given by

$$f'(x) = \frac{d}{dx}[\ln(1+x^2)] = \ln'(1+x^2)\cdot\frac{d}{dx}[1+x^2]. \tag{7.8}$$

Then, using the formula in (7.7) for the derivative of the natural logarithm function, we obtain from (7.8) that

$$f'(x) = \frac{1}{1+x^2}\cdot(2x) = \frac{2x}{1+x^2}, \qquad \text{for all } x\in\mathbb{R}.$$

7.3 Evaluating Integrals

Let f be a function defined in an open interval, I. Assume that f is continuous on I. For a given $a\in I$, define the area function

$$G(x) = \int_a^x f(t)\ dt, \qquad \text{for all } x\in I. \tag{7.9}$$

It follows from the Second Fundamental Theorem of Calculus (Theorem 7.2.1) that G is differentiable in I and

$$G'(x) = f(x), \qquad \text{for all } x\in I, \tag{7.10}$$

since we are assuming that f is continuous in I. Suppose that we are given a real–valued function, F, defined on I with the property that f is differentiable in I and

$$F'(x) = f(x), \qquad \text{for all } x\in I. \tag{7.11}$$

Consider the difference of the functions G and F,

$$S(x) = G(x) - F(x), \qquad \text{for all } x\in I. \tag{7.12}$$

It follows from (7.12) that the function S is differentiable and

$$S'(x) = G'(x) - F'(x) = f(x) - f(x) = 0, \qquad \text{for all } x\in I, \tag{7.13}$$

where we have used (7.10) and (7.11). Thus, applying the result i Example 7.1.2, S must be constant, so that

$$G(x) - F(x) = c, \qquad \text{for all } x\in I, \tag{7.14}$$

where c is a constant. In view of (7.9) and (7.14), we have therefore shown that, if f is continuous on I, and $F\colon I\to\mathbb{R}$ is a differentiable function with $F' = f$, then, for any $a\in I$,

$$\int_a^x f(t)\ dt = F(x) + c, \qquad \text{for all } x\in I, \tag{7.15}$$

and some constant c. Substituting a for x in (7.15) we obtain that

$$0 = F(a) + c,$$

from which we get the following value for $c = -F(a)$. Substituting this value in (7.15) yields

$$\int_a^x f(t)\ dt = F(x) - F(a), \quad \text{for all } x \in I, \tag{7.16}$$

The expression in (7.16), for the case in which f is continuous in I, is usually referred to as the Fundamental Theorem of Calculus. We will call it the Third Fundamental Theorem of Calculus.

Theorem 7.3.1 (Third Fundamental Theorem of Calculus (FTC III))**.** Let f be a continuous function defined in an open interval I. Assume that there exists a function, F, that is differentiable in I and $F'(x) = f(x)$ for all $x \in I$. Then, for any $a, b \in I$,

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

Appendix A

Facts from Algebra

A.1 The Binomial Theorem

Given a positive integer, n, we define the factorial of n, denoted $n!$ and read "n factorial," by

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

The factorial of 0 is defined to be $0! = 1$.

Given real numbers, a and b, the expansion of $a+b$ raised to the n^{th} power is given by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \tag{A.1}$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are called binomial coefficients.

A.2 Some Factorization Facts

For real numbers a and b,

$$a^2 - b^2 = (a-b)(a+b),$$

and

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

In general, for any positive integer n,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots ab^{n-2} + b^{n-1}).$$

Appendix B

Limits Facts

B.1 Limits of Sequence

We begin with the definition of the limit of a sequence.

Definition B.1.1 (Limit of a Sequence)**.** We say that a sequence (a_n) converges to a limit ℓ if for every $\varepsilon > 0$ we can find positive integer N such that

$$n \geqslant N \text{ implies that } |a_n - \ell| < \varepsilon.$$

We write

$$\lim_{n\to\infty} |a_n - \ell| = 0.$$

or

$$\lim_{n\to\infty} a_n = \ell.$$

We first see that the Definition B.1.1 implies that a convergent sequence can have at most one limit ℓ. For, if there were two numbers, ℓ_1 and ℓ_2, for which the conditions in the definition are satisfies; that is, for given $\varepsilon > 0$ there exist positive integers N_1 and N_2 such that

$$n \geqslant N_1 \text{ implies that } |a_n - \ell_1| < \varepsilon. \tag{B.1}$$

and

$$n \geqslant N_2 \text{ implies that } |a_n - \ell_2| < \varepsilon, \tag{B.2}$$

then, using the triangle inequality (see (3.5)), we get that

$$|\ell_1 - \ell_2| \leqslant |\ell_1 - a_n| + |a_n - \ell_2|, \quad \text{for all } n. \tag{B.3}$$

Thus, if we let n denote the larger of N_1 and N_2, we get from (B.1), (B.2) and (B.3) that

$$|\ell_1 - \ell_2| < 2\varepsilon \tag{B.4}$$

Since $\varepsilon > 0$ is arbitrary in (B.4), we conclude from (B.4) that $|\ell_1 - \ell_2| = 0$, which says that $\ell_1 = \ell_2$. Otherwise, if $\ell_1 \neq \ell_2$, then $|\ell_1 - \ell_2| > 0$. We can therefore set

$$\varepsilon = \frac{|\ell_1 - \ell_2|}{2} > 0. \tag{B.5}$$

Now, since (B.4) holds true for any positive value of ε, it must also hold true for the particular value in (B.5); thus, it must be the case that

$$|\ell_1 - \ell_2| < |\ell_1 - \ell_2|,$$

which is nonsense. Hence, $\ell_1 = \ell_2$; therefore, if a limit exists, it must be unique.

Appendix C

Properties of Continuous Functions

Appendix D

Properties of the Riemann Integral

Bibliography