Assignment #18

Due on Friday, November 15, 2013

Read Section 7.1 on the *Definition of Convergence in Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 7.2 on the *mgf Convergence Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.4 on *The Poisson Distribution* in DeGroot and Schervish.

Read Section 5.6 on *The Normal Distribution* in DeGroot and Schervish.

Background and Definitions

Definition (Convergence in Distribution)**.** Let (X_n) be a sequence of random variables with cumulative distribution functions F_{X_n}, for $n = 1, 2, 3, \ldots$, and Y be a random variable with cdf F_Y. We say that the sequence (X_n) converges to Y in distribution, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$$

for all x where F_Y is continuous. The distribution of Y is usually called the **limiting distribution** of the sequence (X_n).

Theorem (mgf Convergence Theorem)**.** *Let (X_n) be a sequence of random variables with moment generating functions $\psi_{X_n}(t)$, for $|t| < h$, $n = 1, 2, 3, \ldots$, and some positive number h. Suppose Y has mgf $\psi_Y(t)$ which exists for $|t| < h$. Then, if*

$$\lim_{n\to\infty} \psi_{X_n}(t) = \psi_Y(t), \quad \text{for } |t| < h,$$

it follows that $\lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$ for all x where F_Y is continuous.

Do the following problems

1. Let a denote a real number and X_a be a discrete random variable with pmf

$$p_{X_a}(x) = \begin{cases} 1 & \text{if } x = a; \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Compute the cdf for X_a and sketch its graph.

   (b) Compute the mgf for X_a and determine $E(X_a)$ and $\text{var}(X_a)$.

2. Let (X_k) denote a sequence of independent identically distributed random variables such that $X_k \sim \text{Normal}(\mu, \sigma^2)$ for every $k = 1, 2, \ldots$, and for some $\mu \in \mathbb{R}$ and $\sigma > 0$. For each $n \geqslant 1$, define

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

   (a) Determine the mgf, $\psi_{\overline{X}_n}(t)$, for $\overline{X}_n$, and compute $\lim_{n\to\infty} \psi_{\overline{X}_n}(t)$.

   (b) Find the limiting distribution of $\overline{X}_n$ as $n \to \infty$. (*Hint:* Compare your answer in part (a) to your answer in part (b) of problem 1.)

3. Let (X_k) and $\overline{X}_n$ be defined as in the previous problem. Define $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ for all $n \geqslant 1$.

   (a) Determine the mgf, $\psi_{Z_n}(t)$, for Z_n, and compute $\lim_{n\to\infty} \psi_{Z_n}(t)$.

   (b) Find the limiting distribution of Z_n as $n \to \infty$.

4. Let (Y_n) be a sequence of discrete random variables having pmfs

$$p_{Y_n}(y) = \begin{cases} 1 & \text{if } y = n, \\ 0 & \text{elsewhere.} \end{cases}$$

   Compute the mgf of Y_n for each $n = 1, 2, 3, \ldots$

   Does $\lim_{n\to\infty} \psi_{Y_n}(t)$ exist for any t in an open interval around 0?

   Does the sequence (Y_n) have a limiting distribution? Justify your answer.

5. Let $q = 0.95$ denote the probability that a person, in certain age group, lives at least 5 years.

   (a) If we observe 60 people from that group and assume independence, what is the probability that at least 56 of them live 5 years or more?

   (b) Find and approximation to the result of part (a) using the Poisson distribution.