Assignment #22

Due on Monday, November 25, 2013

Read Chapter 8 on *Introduction to Estimation* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 8.2 on *The Chi–Square Distribution* in DeGroot and Schervish.

Read Section 8.3 on the *Joint Distribution of the Sample Mean and the Variance* in DeGroot and Schervish.

1. Let (X_k) denote a sequence of independent, identically distributed Normal(μ,σ^2) random variables. In this problem we consider two ways of estimating the variance σ^2 based on random samples of size n, $X_1, X_2, \ldots, X_n$.

   (a) We can estimate σ^2 by using the estimator $\widehat{\sigma}_n^2 = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}(X_k - \overline{X}_n)^2$.

   The estimator $\widehat{\sigma}_n^2$ is called the maximum likelihood estimator for σ^2. Compute $E(\widehat{\sigma}_n^2)$. Is $\widehat{\sigma}_n^2$ an unbiased estimator for σ^2?

   (b) The sample variance, S_n^2, is defined by $S_n^2 = \dfrac{1}{n-1}\displaystyle\sum_{k=1}^{n}(X_k - \overline{X}_n)^2$.

   Compute $E(S_n^2)$. Is $\widehat{\sigma}_n^2$ an unbiased estimator for σ^2?

2. **The Gamma Function.** The gamma function, $\Gamma(x)$, plays a very important role in the definitions a several probability distributions which are very useful in applications. It is defined as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \quad \text{for all} \quad x > 0. \tag{1}$$

   Note: $\Gamma(x)$ can also be defined for negative values of x which are not integers; it is not defined at $x = 0$. In this course, we will only consider $\Gamma(x)$ for $x > 0$.

   Derive the following identities:

   (a) $\Gamma(1) = 1$.

   (b) $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

   (c) $\Gamma(n+1) = n!$ for all non–negative integers n.

3. Let $\Gamma\colon (0,\infty) \to \mathbb{R}$ be as defined in (1).

   (a) Compute $\Gamma(1/2)$.
   Hint: The change of variable $t = z^2/2$ might come in handy. Recall that if $Z \sim \text{Normal}(0,1)$, then its pdf is given by

   $$f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{for all } z \in \mathbb{R}.$$

   (b) Compute $\Gamma(3/2)$.

4. Use the results of Problems 2 and 3 to derive the identity:

   $$\Gamma\left(\frac{k}{2}\right) = \frac{\Gamma(k)\sqrt{\pi}}{2^{k-1}\,\Gamma\left(\frac{k+1}{2}\right)}$$

   for every positive, odd integer k.

   Suggestion: Proceed by induction on k.

5. Let α and β denote positive real numbers and define $f(x) = Cx^{\alpha-1}e^{-x/\beta}$ for $x > 0$ and $f(x) = 0$ for $x \leqslant 0$, where C denotes a positive real number.

   (a) Find the value of C so that f is the pdf for some distribution.

   (b) For the value of C found in part (a), let f denote the pdf of a random variable X. Compute the mgf of X.

   Hint: The pdf found in part (a) is related to the Gamma function.