Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered $1, 2, 3, 4, 5$ respectively, and the blue chips are numbered $1, 2, 3$ respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

3. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually exclusive events in $\mathcal{B}$. Find $\Pr[(E_1 \cup E_2) \cap E_3]$ and $\Pr(E_1^c \cup E_2^c)$.

4. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B events in $\mathcal{B}$. Show that $\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B)$.

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with probabilities $\dfrac{1}{2}$, $\dfrac{1}{3}$ and $\dfrac{1}{4}$, respectively. Compute the exact value of $\Pr(E_1 \cup E_2 \cup E_3)$.

6. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with $\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \dfrac{1}{4}$. Compute $\Pr[(E_1^c \cap E_2^c) \cup E_3]$.

7. A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it.

   (a) If a part passes the inspection, what is the probability that is is a good part?

   (b) Given that a part passes the inspection, what is the probability that is is slightly defective?

   (c) Assume that a one-year warranty is given for the parts that are shipped to customers. Suppose that a good part fails within the first year with probability 0.01, while a slightly defective part fails within the first year with probability 0.10. What is the probability that a customer receives a part that fails within the first year and is therefore entitled to a warranty replacement?

8. Toss a fair coin three times in a row. Let A denote the event that either the three tosses yield three heads or three tails; B the event that at least two heads come up; and C the event that at most two tails come up. Out of the pairs of events: (A, B), (A, C), and (B, C), determine the ones that are independent and the ones that are dependent. Explain your reasoning.

9. A bowl contains 10 chips of the same size and shape. One and only one of these chips is red. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn. Let X denote the number of draws needed to get the red chip. Determine the pmf of X and compute $\Pr(X \leq 4)$.

10. Let X have pmf given by $p_X(x) = \dfrac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

11. Let X have pmf given by $p_X(x) = \left(\frac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$ and $p_X(x) = 0$ elsewhere. Give the pmf of $Y = X^3$.

12. Let $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty; \\ 0 & \text{if } x \leq 1, \end{cases}$ and define a probability on the Borel σ–field of the real line $\mathbb{R}$ by $\Pr[(a, b)] = \displaystyle\int_a^b f(x)\, dx$, for all intervals, (a, b).

    If E_1 denote the interval $(1, 2)$ and E_2 the interval $(4, 5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

13. A *mode* of a distribution of a random discrete variable X is a value of x that maximizes the pmf of X. If there is only one such value, it is called *the mode of the distribution.*

    Let X have pmf given by $p(x) = \left(\dfrac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$, and $p(x) = 0$ elsewhere. Compute a mode of the distribution.

14. Let $f(x) = \begin{cases} cx(1 - x), & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere}, \end{cases}$ where c is a positive constant.

    (a) Determine the value of c so that $\Pr[(a, b)] = \displaystyle\int_a^b f(x)\, dx$, for all intervals, (a, b), defines a probability on the Borel σ–field of the real line $\mathbb{R}$.

    (b) For each $x \in \mathbb{R}$, define $F(x) = \Pr[(-\infty, x]]$. Compute F and sketch its graph. Find the value of x for which $F(x) = 0.5$.