Solutions to Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered $1, 2, 3, 4, 5$ respectively, and the blue chips are numbered $1, 2, 3$ respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

   ***Solution**:* Let R denote the event that the two chips are red. Then the assumption that the chips are drawn at random and without replacement implies that

$$\Pr(R) = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{5}{14}.$$

   Similarly, if B denotes the event that both chips are blue, then

$$\Pr(B) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}.$$

   It then follows that the probability that both chips are of the same color is

$$\Pr(R \cup B) = \Pr(R) + \Pr(B) = \frac{13}{28},$$

   since R and B are mutually exclusive.

   Let N denote the event that both chips show the same number. Then,

$$\Pr(N) = \frac{3}{\binom{8}{2}} = \frac{3}{28}.$$

   Finally, since $R \cup B$ and N are mutually exclusive, then the probability that the chips are have either the same number or the same color is

$$\Pr(R \cup B \cup N) = \Pr(R \cup B) + \Pr(N) = \frac{13}{28} + \frac{3}{28} = \frac{16}{28} = \frac{2}{7}.$$

   □

2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

***Solution**:* Let N denote the event that the person will not win any prize. Then

$$\Pr(N) = \frac{\binom{995}{10}}{\binom{1000}{10}}; \tag{1}$$

that is, the probability of purchasing 10 non–winning tickets.

It follows from (1) that

$$\begin{aligned} \Pr(N) &= \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} \\ &= \frac{435841667261}{458349513900} \qquad (2) \\ &\approx 0.9509. \end{aligned}$$

Thus, using the result in (2), the probability of the person winning at least one of the prizes is

$$\begin{aligned} \Pr(N^c) &= 1 - \Pr(N) \\ &\approx 1 - 0.9509 \\ &= 0.0491, \end{aligned}$$

or about 4.91%. □

3. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually exclusive events in $\mathcal{B}$. Find $\Pr[(E_1 \cup E_2) \cap E_3]$ and $\Pr(E_1^c \cup E_2^c)$.

   ***Solution**:* Since E_1, E_2 and E_3 are mutually disjoint events, it follows that $(E_1 \cup E_2) \cap E_3 = \emptyset$; so that

   $$\Pr[(E_1 \cup E_2) \cap E_3] = 0.$$

   Next, use De Morgan's law to compute

   $$\begin{aligned} \Pr(E_1^c \cup E_2^c) &= \Pr([E_1 \cap E_2]^c) \\ &= \Pr(\emptyset^c) \\ &= \Pr(\mathcal{C}) \\ &= 1. \end{aligned}$$

□

4. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B events in $\mathcal{B}$. Show that
$$\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B). \tag{3}$$

   Solution: Since $A \cap B \subseteq A$, it follows that
$$\Pr(A \cap B) \leqslant \Pr(A). \tag{4}$$

   Similarly, since $A \subseteq A \cup B$, we get that
$$\Pr(A) \leqslant \Pr(A \cup B). \tag{5}$$

   Next, use the identity
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$
   and fact that that
$$\Pr(A \cap B) \geqslant 0,$$
   to obtain that
$$\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B). \tag{6}$$

   Finally, combine (4), (5) and (6) to obtain (3). □

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively. Compute the exact value of $\Pr(E_1 \cup E_2 \cup E_3)$.

   Solution: First, use De Morgan's law to compute
$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c \cap E_2^c \cap E_3^c) \tag{7}$$

   Then, since E_1, E_2 and E_3 are mutually independent events, it follows from (7) that
$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c) \cdot \Pr(E_2^c) \cdot \Pr(E_3^c),$$
   so that
$$\begin{aligned}\Pr[(E_1 \cup E_2 \cup E_3)^c] &= (1 - \Pr(E_1))(1 - \Pr(E_2))(1 - \Pr(E_3)) \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4},\end{aligned}$$

so that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{1}{4}. \tag{8}$$

It then follows from (8) that

$$\Pr(E_1 \cup E_2 \cup E_3) = 1 - \Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{3}{4}.$$

□

6. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with $\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \frac{1}{4}$. Compute $\Pr[(E_1^c \cap E_2^c) \cup E_3]$.

   ***Solution**:* First, use De Morgan's law to compute

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c \cap E_3^c] \tag{9}$$

   Next, use the assumption that E_1, E_2 and E_3 are mutually independent events to obtain from (9) that

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c] \cdot \Pr[E_3^c], \tag{10}$$

   where

$$\Pr[E_3^c] = 1 - \Pr(E_3) = \frac{3}{4}, \tag{11}$$

   and

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - \Pr[E_1^c \cap E_2^c] \\ &= 1 - \Pr[E_1^c] \cdot \Pr[E_2^c], \end{aligned} \tag{12}$$

   by the independence of E_1 and E_2.

   It follows from the calculations in (12) that

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - (1 - \Pr[E_1])(1 - \Pr[E_2]) \\ &= 1 - \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \\ &= 1 - \frac{3}{4} \cdot \frac{3}{4} \\ &= \frac{7}{16} \end{aligned} \tag{13}$$

Substitute (11) and the result of the calculations in (13) into (10) to obtain

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \frac{7}{16} \cdot \frac{3}{4} = \frac{21}{64}. \tag{14}$$

Finally, use the result in (14) to compute

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c) \cup E_3^c] &= 1 - \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \\ &= 1 - \frac{21}{64} \\ &= \frac{43}{64}. \end{aligned}$$

□

7. A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it.

   (a) If a part passes the inspection, what is the probability that is is a good part?

   ***Solution**:* Let G denote the event that the machine produces a good part, S denote the event that the machine produces a slightly defective part, and D the event that the machine produces an obviously defective part. We are then given that

   $$\Pr(G) = 0.90, \quad \Pr(S) = 0.02 \quad \text{and} \quad \Pr(D) = 0.08.$$

   A part passes inspection if it is good part or if it is slightly defective; in other words, if the complement of event D occurs (note that $D^c = G \cup S$). Thus, the probability that a part is good, given that it passed inspection is the conditional probability

   $$\begin{aligned} \Pr(G \mid D^c) &= \frac{\Pr(G \cap D^c)}{\Pr(D^c)} \\ &= \frac{\Pr(G)}{\Pr(G \cup S)} \\ &= \frac{0.90}{0.92} \\ &= \frac{45}{46}. \end{aligned}$$

□

(b) Given that a part passes the inspection, what is the probability that it is slightly defective?

Solution: In this case we compute the conditional probability

$$\begin{aligned}\Pr(S \mid D^c) &= \frac{\Pr(S \cap D^c)}{\Pr(D^c)} \\ &= \frac{\Pr(S)}{\Pr(G \cup S)} \\ &= \frac{0.02}{0.92} \\ &= \frac{1}{46}\end{aligned}$$

□

(c) Assume that a one-year warranty is given for the parts that are shipped to customers. Suppose that a good part fails within the first year with probability 0.01, while a slightly defective part fails within the first year with probability 0.10. What is the probability that a customer receives a part that fails within the first year and is therefore entitled to a warranty replacement?

Solution: Let F denote the event that a part that has passed inspection will fail within the first year after shipping. Let G_I denote the event that a good part has passed inspection and been shipped. From part (a) we have that

$$\Pr(G_I) = \Pr(G \mid D^c) = \frac{45}{46}.$$

Similarly, denoting by S_I the event that a slightly defective part has passed inspection, we have from part (b) that

$$\Pr(S_I) = \Pr(S \mid D^c) = \frac{1}{46}.$$

We are given that

$$\Pr(F \mid G_I) = 0.01 \quad \text{and} \quad \Pr(F \mid S_I) = 0.10.$$

It then follows from the Law of Total Probability that

$$\begin{aligned}\Pr(F) &= \Pr(G_I)\cdot\Pr(F\mid G_I)+\Pr(S_I)\cdot\Pr(F\mid S_I)\\ &= \frac{45}{46}\cdot(0.01)+\frac{1}{46}\cdot(0.10)\\ &\doteq 0.0112.\end{aligned}$$

Thus, the probability that a customer receives a part that fails within the first year is about 1.12%. □

8. Toss a fair coin three times in a row. Let A denote the event that either the three tosses yield three heads or three tails; B the event that at least two heads come up; and C the event that at most two tails come up. Out of the pairs of events: (A,B), (A,C), and (B,C), determine the ones that are independent and the ones that are dependent. Explain your reasoning.

   ***Solution**:* The sample space for this experiment is

$$\mathcal{C}=\{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT\}.$$

The events A, B and C are

$$A=\{HHH,TTT\},$$

$$B=\{HHH,HHT,HTH,THH\},$$

and

$$C=\{HHH,HHT,HTH,HTT,THH,THT,TTH\},$$

respectively. Since all the elements of $\mathcal{C}$ are equally likely, it follows that

$$\Pr(A)=\frac{1}{4},\quad \Pr(B)=\frac{1}{2},\quad \text{and}\quad \Pr(C)=\frac{7}{8}.$$

Note that $A\cap B=\{HHH\}$, so that

$$\Pr(A\cap B)=\frac{1}{8}=\frac{1}{4}\cdot\frac{1}{2}=\Pr(A)\cdot\Pr(B);$$

thus, A and B are independent.

Next, compute $A\cap C=\{HHH\}$, so that

$$\Pr(A\cap C)=\frac{1}{8}\neq\frac{1}{4}\cdot\frac{7}{8}=\Pr(A)\cdot\Pr(C);$$

thus, A and C are not independent.

Finally, compute $B \cap C = \{HHH, HHT, HTH, THH\}$, so that

$$\Pr(B \cap C) = \frac{1}{2} \neq \frac{1}{2} \cdot \frac{7}{8} = \Pr(B) \cdot \Pr(C);$$

thus, B and C are not independent. □

9. A bowl contains 10 chips of the same size and shape. One and only one of these chips is red. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn. Let X denote the number of draws needed to get the red chip. Determine the pmf of X and compute $\Pr(X \leq 4)$.

   Solution: Compute

$$\begin{aligned} \Pr(X = 1) &= \frac{1}{10} \\ \Pr(X = 2) &= \frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10} \\ \Pr(X = 3) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10} \\ &\vdots \\ \Pr(X = 10) &= \frac{1}{10} \end{aligned}$$

   Thus,

$$p_X(k) = \begin{cases} \dfrac{1}{10} & \text{for } k = 1, 2, \ldots, 10; \\ \\ 0 & \text{elsewhere.} \end{cases} \qquad (15)$$

   Next, use (15) to compute

$$\Pr(X \leqslant 4) = \sum_{k=1}^{4} p_X(k) = \frac{4}{10} = \frac{2}{5}.$$

□

10. Let X have pmf given by $p_X(x) = \dfrac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

***Solution**:* Note that the possible values for Y are 3, 5 and 7

Compute

$$\Pr(Y = 3) = \Pr(2X + 1 = 3) = \Pr(X = 1) = \frac{1}{3}.$$

Similarly, we get that

$$\Pr(Y = 5) = \Pr(X = 2) = \frac{1}{3},$$

and

$$\Pr(Y = 7) = \Pr(X = 3) = \frac{1}{3}.$$

Thus,

$$p_Y(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 3, 5, 7; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

11. Let X have pmf given by $p_X(x) = \left(\frac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$ and $p_X(x) = 0$ elsewhere. Give the pmf of $Y = X^3$.

    ***Solution**:* Compute, for $y = k^3$, for $k = 1, 2, 3, \ldots$,

    $$\Pr(Y = y) = \Pr(X^3 = k^3) = \Pr(X = k) = \left(\frac{1}{2}\right)^k,$$

    so that

    $$\Pr(Y = y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \quad \text{for } y = k^3, \text{ for some } k = 1, 2, 3, \ldots$$

    Thus,

    $$p_Y(y) = \begin{cases} \left(\dfrac{1}{2}\right)^{y^{1/3}}, & \text{for } y = k^3, \text{ for some } k = 1, 2, 3, \ldots; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

12. Let $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty; \\ 0 & \text{if } x \leq 1, \end{cases}$ and define a probability on the Borel σ–field of the real line $\mathbb{R}$ by $\Pr[(a,b)] = \int_a^b f(x)\, dx$, for all intervals, (a,b).

    If E_1 denote the interval $(1,2)$ and E_2 the interval $(4,5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

    Solution: Compute

$$\Pr(E_1) = \int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^2 = \frac{1}{2},$$

$$\Pr(E_2) = \int_4^5 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_4^5 = \frac{1}{20},$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20},$$

    since E_1 and E_2 are mutually exclusive, and

$$\Pr(E_1 \cap E_2) = 0,$$

    since E_1 and E_2 are mutually exclusive. □

13. A *mode* of a distribution of a random discrete variable X is a value of x that maximizes the pmf of X. If there is only one such value, it is called *the mode of the distribution.*

    Let X have pmf given by $p(x) = \left(\dfrac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$, and $p(x) = 0$ elsewhere. Compute a mode of the distribution.

    Solution: Note that $p(x)$ is decreasing; so, $p(x)$ is maximized when $x = 1$. Thus, 1 is the mode of the distribution of X. □

14. Let $f(x) = \begin{cases} cx(1-x), & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere}, \end{cases}$ where c is a positive constant.

    (a) Determine the value of c so that $\Pr[(a,b)] = \int_a^b f(x)\, dx$, for all intervals, (a,b), defines a probability on the Borel σ–field of the real line $\mathbb{R}$.

Solution: We choose c so that $\Pr(\mathbb{R}) = 1$, where

$$\begin{aligned}
\Pr(\mathbb{R}) &= \int_{-\infty}^{\infty} f(x)\, dx \\
&= \int_0^1 cx(1-x)\, dx \\
&= c\int_0^1 [x - x^2]\, dx \\
&= c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\
&= \frac{c}{6}.
\end{aligned}$$

It then follows that $c = 6$. □

(b) For each $x \in \mathbb{R}$, define $F(x) = \Pr[(-\infty, x]]$. Compute F and sketch its graph. Find the value of x for which $F(x) = 0.5$.

Solution: We compute $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\, dt$, for $x \in \mathbb{R}$, where

$$f(t) = \begin{cases} 6t(1-t), & \text{if } 0 < t < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

If $x \leqslant 0$ we have that $f(t) = 0$ for all $t \leqslant x$, so that

$$F(x) = 0, \quad \text{for } x \leqslant 0.$$

If $0 < x \leqslant 1$, we get that

$$\begin{aligned}
F(x) &= \int_0^x 6t(1-t)\, dt \\
&= 6\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^x \\
&= 3x^2 - 2x^3.
\end{aligned}$$

Finally, if $x \geqslant 1$, we have that

$$F(x) = 1.$$

Putting all these calculations together we get

$$F(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ 3x^2 - 2x^3, & \text{if } 0 < x \leqslant 1; \\ 1, & \text{if } x > 1. \end{cases}$$

A sketch of the graph of F is found in Figure 1.

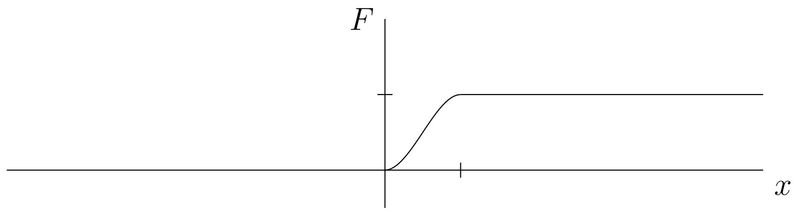

Figure 1: Sketch of $F(x)$ in Problem 14

Observe that $F(0.5) = 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$, so that $x = \frac{1}{2}$ is the unique value of x for which $F(x) = 0.5$. $\square$