Solutions to Exam 1

1. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A, B and C be events in $\mathcal{B}$.

   (a) Given that $\Pr(A \cap B) = 0.3$ and $\Pr(A \cap B^c) = 0.1$, compute $\Pr(A)$. Explain your reasoning.

   Solution: Observe that $A = (A \cap B) \cup (A \cap B^c)$, where $A \cap B$ and $A \cap B^c$ are disjoint. We then have that

   $$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c) = 0.3 + 0.1 = 0.4.$$

   □

   (b) Assume A and B are independent with $\Pr(A) = 0.25$ and $\Pr(B) = 0.75$. Compute $\Pr(A \backslash B)$. Explain your reasoning.

   Solution: Note that $A \backslash B = A \cap B^c$, where A and B^c are independent, so that

   $$\Pr(A \backslash B) = \Pr(A) \cdot \Pr(B^c) = \Pr(A) \cdot (1 - \Pr(B)).$$

   Thus,

   $$\Pr(A \backslash B) = (0.25) \cdot (1 - (0.75)) = (0.25)^2 = 0.0625.$$

   □

   (c) Define what it means for A, B, C to be mutually independent.

   Solution: A, B, C to be mutually independent means that

   - $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$,
   - $\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$,
   - $\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$, and
   - $\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$.

   □

   (d) Assume that A, B and C are mutually independent with $\Pr(A) = \Pr(B) = \Pr(C) = 1/3$. Compute $\Pr(A \cup B \cup C)$. Explain your reasoning.

   Solution: Compute

   $$\begin{aligned} \Pr(A \cup B \cup C) &= 1 - \Pr[(A \cup B \cup C)^c] \\ &= 1 - \Pr[A^c \cap B^c \cap C^c], \end{aligned}$$

where we have used DeMorgan's Law. Thus, using independence,

$$\begin{aligned}
\Pr(A \cup B \cup C) &= 1 - \Pr(A^c) \cdot \Pr(B^c) \cdot \Pr(C^c) \\
&= 1 - (1 - \Pr(A)) \cdot (1 - \Pr(B)) \cdot (1 - \Pr(C)) \\
&= 1 - \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{3}\right) \\
&= 1 - \frac{8}{27} \\
&= \frac{19}{27}.
\end{aligned}$$

□

2. A box contains four red balls and three blue balls. Suppose that three of the balls are selected at random and without replacement.

   (a) Compute the probability that all three of the selected balls are blue.

   ***Solution**:* Let B_3 denote the event that all three balls are blue. Then,

   $$\Pr(B_3) = \frac{\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}.$$

   □

   (b) Compute the probability that the selected group of three balls contains at least one of the red balls.

   ***Solution**:* Note that B_3 is the same as the event of no red balls in the group of three. Hence, the the event that there is at least one red ball in the group of three is the complement of B_3; thus, the probability there is ate least one red ball in the selected group of three is

   $$\Pr(B_3^c) = 1 - \Pr(B_3) = 1 - \frac{1}{35} = \frac{34}{35}.$$

   □

3. I have two coins in my pocket; one is a fair coin and the other is a two–headed coin. I grab one of the coins at random and then flip it.

(a) Compute the probability that the flip will yield a head.

***Solution**:* Let H denote the event that the flip yields a head, F the event that the flipped coin is the fair one, and T the event that the flipped coin is the trick coin. Then, by the Law of Total Probability,

$$\Pr(H) = \Pr(F)\Pr(H \mid F) + \Pr(T)\Pr(H \mid T),$$

where $\Pr(F) = \dfrac{1}{2}$, $\Pr(T) = \dfrac{1}{2}$, $\Pr(H \mid F) = \dfrac{1}{2}$, and $\Pr(H \mid T) = 1$. Thus,

$$\Pr(H) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}. \tag{1}$$

$\square$

(b) If the flipped coin shows a head, what is the probability that it is the fair coin?

***Solution**:* We compute the conditional probability

$$\Pr(F \mid H) = \frac{\Pr(F \cap H)}{\Pr(H)}, \tag{2}$$

where

$$\Pr(F \cap H) = \Pr(F)\Pr(H \mid F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \tag{3}$$

Substituting the results of (1) and (3) into (2) yields

$$\Pr(F \mid H) = \frac{1}{3}.$$

$\square$

4. A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. Assume that the probability that one or more accidents will occur during any given month is $3/5$, and that the number of accidents that occur in any given month is independent of the number of accidents that occur in all other months. Let X denote the number of months until at least one accident occurs; e.g., $X = 1$ if at least one accident occurs in the first month; $X = 2$ if no accident occurs in the first month, but at least one accident occurs in the second month; etc.

   (a) Compute the pmf of X.

Solution:

$$p_{_X}(k) = \begin{cases} \left(\dfrac{2}{5}\right)^{k-1} \cdot \dfrac{3}{5}, & \text{for } k = 1, 2, \ldots; \\ \\ 0, & \text{elsewhere.} \end{cases} \tag{4}$$

□

(b) Calculate the probability that there will be at least three consecutive months in which no accidents occur.

Solution: We want $\Pr(X \geqslant 4)$, which is given by

$$\Pr(X \geqslant 4) = 1 - \Pr(X < 4),$$

or

$$\Pr(X \geqslant 4) = 1 - \Pr(X \leqslant 3), \tag{5}$$

where

$$\Pr(X \leqslant 3) = p_{_X}(1) + p_{_X}(2) + p_{_X}(3). \tag{6}$$

Using the formula for the pmf for X in (4) we obtain from (6) that

$$\begin{aligned} \Pr(X \leqslant 3) &= \frac{3}{5} + \frac{2}{5} \cdot \frac{3}{5} + \frac{4}{25} \cdot \frac{3}{5} \\ &= \frac{3}{5} + \frac{6}{25} + \frac{12}{125} \\ &= \frac{75 + 30 + 12}{125} \end{aligned}$$

so that

$$\Pr(X \leqslant 3) = \frac{117}{125} \tag{7}$$

Substituting the result of (7) into (5) then yields

$$\Pr(X \geqslant 4) = \frac{8}{125}.$$

□