

Assignment #2

Due on Monday, September 15, 2014

Read Sections 2.1 and 2.3 on σ -fields in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Sections 2.2 on *Some Set Algebra* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 1.4 on *Set Theory* in DeGroot and Schervish.

Do the following problems

Background and Definitions

- A σ -field, \mathcal{B} , is a collection of subsets of a sample space \mathcal{C} , referred to as **events**, which satisfy:
 - (1) $\emptyset \in \mathcal{B}$ (\emptyset denotes the empty set)
 - (2) If $E \in \mathcal{B}$, then its complement, E^c , is also an element of \mathcal{B} .
 - (3) If $(E_1, E_2, E_3 \dots)$ is a sequence of events, then

$$E_1 \cup E_2 \cup E_3 \cup \dots = \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.$$

- Let \mathcal{S} denote a collection of subsets of a sample space \mathcal{C} . The σ -field generated by \mathcal{S} , denoted by $\mathcal{B}(\mathcal{S})$, is the smallest σ -field in \mathcal{C} which contains \mathcal{S} .
- \mathcal{B}_o denotes the Borel σ -field of the real line, \mathbb{R} . This is the σ -field generated by the semi-infinite intervals

$$(-\infty, b], \quad \text{for } b \in \mathbb{R}.$$

1. Let A , B and C be subsets of a sample space \mathcal{C} . Prove the following
 - (a) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
 - (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

2. Let \mathcal{C} be a sample space and \mathcal{B} be a σ -field of subsets of \mathcal{C} . Prove that if $\{E_1, E_2, E_3, \dots\}$ is a sequence of events in \mathcal{B} , then

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$

Hint: Use De Morgan's Laws.

3. Let \mathcal{C} be a sample space and \mathcal{B} be a σ -field of subsets of \mathcal{C} . For fixed $B \in \mathcal{B}$ define the collection of subsets

$$\mathcal{B}_B = \{D \subseteq \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

Show that \mathcal{B}_B is a σ -field.

Note: In this case, the complement of $D \in \mathcal{B}_B$ has to be understood as $B \setminus D$; that is, the complement relative to B . The σ -field \mathcal{B}_B is the σ -field \mathcal{B} restricted to B , or *conditioned on B* .

4. Let \mathcal{S} denote the collection of all bounded, open intervals (a, b) , where a and b are real numbers with $a < b$. Show that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o;$$

that is, the σ -field generated by bounded open intervals is the Borel σ -field.

Hints:

- We have already seen in the lecture that \mathcal{B}_o contains all bounded open intervals.
 - Observe also that the semi-infinite open interval (b, ∞) can be expressed as the union of the sequence of bounded intervals (b, k) , for $k = 1, 2, 3, \dots$
5. Show that for every real number a , the singleton $\{a\}$ is in the Borel σ -field \mathcal{B}_o .
- Hint:* Express $\{a\}$ as an intersection of a sequence of open intervals.