Solutions to Exam 1 (Part I)

1. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B denote events in $\mathcal{B}$.

(a) State what it means for A and B to be independent.

Answer: The events A and B are independent means that

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

□

(b) State what it means for A and B to be mutually exclusive.

Answer: The events A and B are mutually exclusive means that

$$A \cap B = \emptyset.$$

□

(c) Assume that $\Pr(B) > 0$. Define the conditional probability of A given B.

Answer: The conditional probability of A given B, $\Pr(A \mid B)$, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

□

(d) Given that $\Pr(B) > 0$, state the multiplication rule for computing the probability of the joint occurrence of A and B.

Answer:

$$\Pr(A \cap B) = \Pr(B) \cdot \Pr(A \mid B).$$

□

(e) State the inclusion–exclusion principle for computing $\Pr(A \cup B)$.

Answer:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

□

2. An experiment consists of flipping a fair coin three consecutive times.

   (a) List all the elements of the sample space, $\mathcal{C}$, for this experiment.

   Solution:

$$\mathcal{C}: \begin{cases} HHH \\ HHT \\ HTH \\ HTT \\ THH \\ THT \\ TTH \\ TTT \end{cases}$$

□

   (b) For each element, c, of the sample space, $\mathcal{C}$, let $N_H(c)$ denote the number of heads in c, and $N_T(c)$ the number of tails in c. Put

$$X(c) = N_H(c) - N_T(c), \quad \text{for all } c \in \mathcal{C}.$$

   List all possible values for the random variable X.

   Solution: Table 1 shows the values of X for each element of $\mathcal{C}$. Thus, the

$\mathcal{C}$	X
HHH	3
HHT	1
HTH	1
HTT	-1
THH	1
THT	-1
TTH	-1
TTT	-3

Table 1: Values of X

   possible values of X are: -3, -1, 1 and 3. □

   (c) Compute the probability mass function (pmf) for X. Explain the reasoning behind your calculations.

   Solution: Since, we are assuming that the coin is fair, each of the outcomes in the first column in Table 1 has the same likelihood; namely, $\Pr(\{c\}) = 1/8$ for each $c \in \mathcal{C}$.

In order to compute the pmf of X, first note that, in view of Table 1,

$$\begin{aligned} (X = -3) &= \{TTT\}; \\ (X = -1) &= \{HTT, THT, TTH\}; \\ (X = 1) &= \{HHT, HTH, THH\}; \\ (X = 3) &= \{HHH\}, \end{aligned}$$

from which we get that

$$\begin{aligned} \Pr(X = -3) &= 1/8; \\ \Pr(X = -1) &= 3/8; \\ \Pr(X = 1) &= 3/8; \\ \Pr(X = 3) &= 1/8. \end{aligned}$$

Thus, the pmf of X is given by

$$p_X(k) = \begin{cases} 1/8, & \text{if } k = -3; \\ 3/8, & \text{if } k = -1; \\ 3/8, & \text{if } k = 1; \\ 1/8, & \text{if } k = 3; \\ 0, & \text{elsewhere.} \end{cases} \tag{1}$$

□

(d) Compute $\Pr(X \leqslant 0)$. Explain the reasoning behind your calculations.
***Solution**:* Using the pmf in (1) we get that

$$\Pr(X \leqslant 0) = p_X(-3) + p_X(-1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}.$$

□