Solutions to Exam 1 (Part II)

1. A point (x, y) is to be selected at random from a square S containing all the points (x, y) such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Suppose that the probability that the selected point will belong to each specified subset of S is equal to the area of that subset, whenever that area of the subset is defined.

   Define the following events:

$$E_1 = \{(x,y) \in S \mid x \leqslant 0.5\},$$
$$E_2 = \{(x,y) \in S \mid y \geqslant 0.5\},$$
and
$$E_3 = \{(x,y) \in S \mid x \leqslant 0.5,\ y \geqslant 0.5\} \cup \{(x,y) \in S \mid x \geqslant 0.5,\ y \leqslant 0.5\}.$$

   (a) Compute $\Pr(E_1)$, $\Pr(E_2)$ and $\Pr(E_3)$.

   Solution: The events E_1, E_2 and E_3 are shown as the shaded regions in Figure 1. In this case, probabilities are given by the areas of the shaded

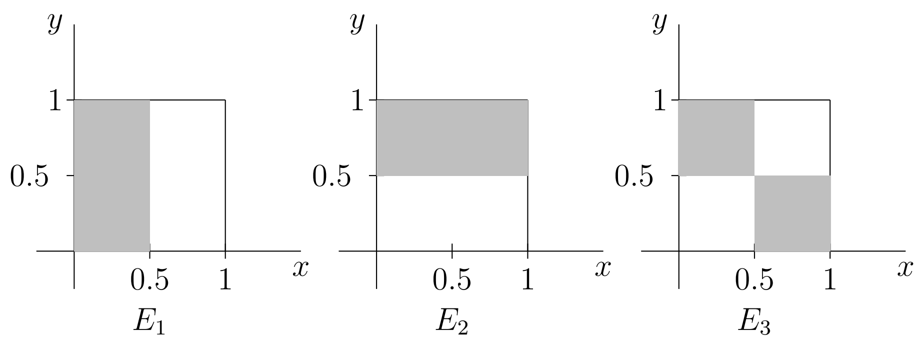

Figure 1: Sketch of Events E_1, E_2 and E_3

   regions; thus,

$$\Pr(E_1) = \text{area}(E_1) = (0.5) \cdot (1) = 0.5,$$
$$\Pr(E_2) = \text{area}(E_2) = (1) \cdot (0.5) = 0.5,$$

   and

$$\Pr(E_3) = \text{area}(E_3) = (0.5) \cdot (0.5) + (0.5) \cdot (0.5) = 0.5.$$

□

   (b) Compute $\Pr(E_1 \cap E_2)$, $\Pr(E_1 \cap E_3)$ and $\Pr(E_2 \cap E_3)$.

   Solution: The events $\Pr(E_1 \cap E_2)$, $\Pr(E_1 \cap E_3)$ and $\Pr(E_2 \cap E_3)$ are shown in Figure 2. Computing the areas of the events in (2), we obtain

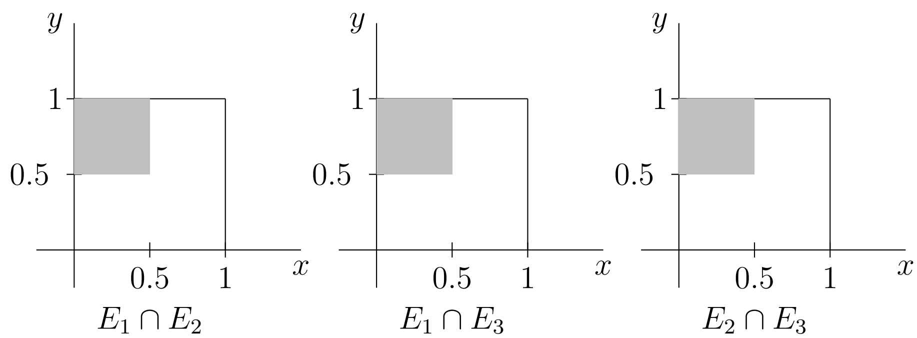

Figure 2: Sketch of Events $E_1 \cap E_2$, $E_1 \cap E_3$ and $E_2 \cap E_3$

$$\Pr(E_1 \cap E_2) = P(E_1 \cap E_3) = \Pr(E_2 \cap E_3) = (0.5) \cdot (0.5) = 0.25. \tag{1}$$

□

(c) Compute $\Pr(E_1 \cap E_2 \cap E_3)$.
***Solution**:* A sketch of the event $E_1 \cap E_2 \cap E_3$ is shown in Figure 3. Compute

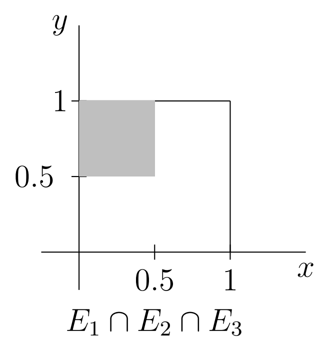

Figure 3: Sketch of Event $E_1 \cap E_2 \cap E_3$

$$\Pr(E_1 \cap E_2 \cap E_3) = \text{area}(E_1 \cap E_2 \cap E_3) = (0.5) \cdot (0.5) = 0.25. \tag{2}$$

□

(d) Compute $\Pr(E_1 \mid E_2 \cap E_3)$.
***Solution**:* Compute

$$\Pr(E_1 \mid E_2 \cap E_3) = \frac{\Pr(E_1 \cap E_2 \cap E_3)}{\Pr(E_2 \cap E_3)} = \frac{0.25}{0.25} = 1,$$

where we have used the results of parts (b) and (c). □

(e) Are the events E_1, E_2 and E_3 pairwise independent? Give reasons for your answer.

***Solution**:* Use the result of part (a) to compute

$$\Pr(E_1) \cdot \Pr(E_2) = (0.5) \cdot (0.5) = 0.25,$$

$$\Pr(E_1) \cdot \Pr(E_3) = (0.5) \cdot (0.5) = 0.25,$$

and

$$\Pr(E_2) \cdot \Pr(E_3) = (0.5) \cdot (0.5) = 0.25.$$

Comparing these with the result in (1) we see that

$$\Pr(E_i \cap E_j) = \Pr(E_i) \cdot \Pr(E_j), \quad \text{for } i \neq j.$$

Hence, E_1, E_2 and E_3 are pairwise independent. □

(f) Are the events E_1, E_2 and E_3 mutually independent? Give reasons for your answer.

***Solution**:* Use the result of part (a) to compute

$$\Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(E_3) = (0.5) \cdot (0.5) \cdot (0.5) = 0.125.$$

Comparing this with the result in (2) we see that

$$\Pr(E_1 \cap E_2 \cap E_3) \neq \Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(E_3);$$

hence, the events E_1, E_2 and E_3 are not mutually independent.

Alternatively, in view of the results in parts (a) and (d), we see that

$$\Pr(E_1 \mid E_2 \cap E_3) \neq \Pr(E_1).$$

□

2. Suppose the probability density function (pdf) of a random variable, X, is as follows:

$$f_X(x) = \begin{cases} c\, e^{-x/\beta}, & \text{for } x \geqslant 0; \\ 0, & \text{elsewhere,} \end{cases}$$

where β is a given positive parameter.

(a) Find the value of c and sketch a graph of the pdf.
***Solution**:* We find c so that

$$\int_{-\infty}^{\infty} f_X(x)\ dx = 1, \tag{3}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\ dx &= \int_0^{\infty} c\ e^{-x/\beta}\ dx \\ &= \lim_{b\to\infty}\int_0^b c\ e^{-x/\beta}\ dx,\end{aligned}$$

or

$$\int_{-\infty}^{\infty} f_X(x)\ dx = c\lim_{b\to\infty}\int_0^b\ e^{-x/\beta}\ dx. \tag{4}$$

In order to evaluate the limit on the right–hand side of (10), we first evaluate the integral

$$\int_0^b\ e^{-x/\beta}\ dx = \left[-\beta e^{-x/\beta}\right]_0^b = \beta - \beta e^{-b/\beta},$$

so that, since $\beta > 0$,

$$\lim_{b\to\infty}\int_0^b\ e^{-x/\beta}\ dx = \beta. \tag{5}$$

Combining (10) and (5) we get

$$\int_{-\infty}^{\infty} f_X(x)\ dx = c\beta. \tag{6}$$

It then follows from (9) and (6) that

$$c = \frac{1}{\beta}.$$

A sketch of the graph of f_X for the case $\beta = 1$ is shown in Figure 4. □

(b) Compute $\Pr(X > \beta)$.
***Solution**:* Compute

$$\begin{aligned}\Pr(X > \beta) &= 1 - \Pr(X \leqslant \beta) \\ &= 1 - \int_0^{\beta}\frac{1}{\beta}e^{-x/\beta}\ dx \\ &= 1 - \left[-e^{-x/\beta}\right]_0^{\beta} \\ &= 1 - \left[1 - e^{-1}\right],\end{aligned}$$

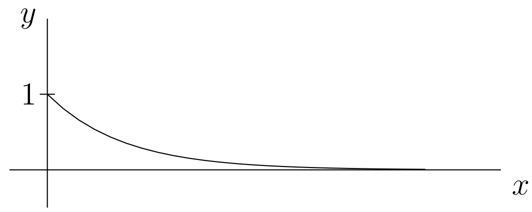

Figure 4: Sketch of graph of $y = f_X(x)$

so that

$$\Pr(X > \beta) = \frac{1}{e}.$$

□

(c) Find a positive value, m, for which

$$\Pr(X \leqslant m) = \frac{1}{2}. \tag{7}$$

Solution: First, compute

$$\begin{aligned} \Pr(X \leqslant m) &= \int_0^m \frac{1}{\beta} e^{-x/\beta}\, dx \\ &= \left[-e^{-x/\beta}\right]_0^m \\ &= 1 - e^{-m/\beta}. \end{aligned}$$

It then follows from (7) that

$$1 - e^{-m/\beta} = \frac{1}{2},$$

or

$$e^{-m/\beta} = \frac{1}{2}. \tag{8}$$

Solving (8) for m then yields

$$m = (\ln 2)\beta.$$

□

3. Suppose that the time, T, that a manufacturing system is out of operation has cumulative distribution function (cdf) given by

$$F_T(t) = \begin{cases} 1 - \left(\dfrac{2}{t}\right)^2, & \text{for } t > 2; \\ \\ 0, & \text{elsewhere.} \end{cases} \tag{9}$$

(a) Assume that t is measured in days. Estimate the probability that the system will be out of operation for at least 4 days.

Solution: Compute

$$\begin{aligned} \Pr(T > 4) &= 1 - \Pr(T \leqslant 4) \\ &= 1 - F_T(4). \end{aligned}$$

Thus, using (9),

$$\Pr(T > 4) = 1 - \left[1 - \left(\frac{2}{4}\right)^2\right] = \frac{1}{4},$$

or 25%. □

(b) Assume that the resulting cost to the company is proportional to $Y = T^2$. Determine the probability density function (pdf) for Y.

Solution: First, compute the cdf of Y:

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y), \quad \text{for } y > 4; \\ &= \Pr(T^2 \leqslant y) \\ &= \Pr(T \leqslant \sqrt{y}); \end{aligned}$$

so that

$$F_Y(y) = F_T(\sqrt{y}), \quad \text{for } y > 4.$$

Thus, using (9) we get that

$$F_Y(y) = \begin{cases} 1 - \dfrac{4}{y}, & \text{for } y > 4; \\ 0, & \text{for } y \leqslant 4. \end{cases} \tag{10}$$

Differentiating (10) with respect to y yields

$$f_{_Y}(y) = \begin{cases} \dfrac{4}{y^2}, & \text{for } y > 4; \\ \\ 0, & \text{for } y \leqslant 4. \end{cases}$$

□