

## Solutions to Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered 1, 2, 3, 4, 5 respectively, and the blue chips are numbered 1, 2, 3 respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

**Solution:** Let  $R$  denote the event that the two chips are red. Then the assumption that the chips are drawn at random and without replacement implies that

$$\Pr(R) = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{5}{14}.$$

Similarly, if  $B$  denotes the event that both chips are blue, then

$$\Pr(B) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}.$$

It then follows that the probability that both chips are of the same color is

$$\Pr(R \cup B) = \Pr(R) + \Pr(B) = \frac{13}{28},$$

since  $R$  and  $B$  are disjoint.

Let  $N$  denote the event that both chips show the same number. Then,

$$\Pr(N) = \frac{3}{\binom{8}{2}} = \frac{3}{28}.$$

Finally, since  $R \cup B$  and  $N$  are disjoint, then the probability that the chips are have either the same number or the same color is

$$\Pr(R \cup B \cup N) = \Pr(R \cup B) + \Pr(N) = \frac{13}{28} + \frac{3}{28} = \frac{16}{28} = \frac{4}{7}.$$

□

2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

**Solution:** Let  $N$  denote the event that the person will not win any prize. Then

$$\Pr(N) = \frac{\binom{995}{10}}{\binom{1000}{10}}; \quad (1)$$

that is, the probability of purchasing 10 non-winning tickets.

It follows from (1) that

$$\begin{aligned} \Pr(N) &= \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} \\ &= \frac{435841667261}{458349513900} \\ &\approx 0.9509. \end{aligned} \quad (2)$$

Thus, using the result in (2), the probability of the person winning at least one of the prizes is

$$\begin{aligned} \Pr(N^c) &= 1 - \Pr(N) \\ &\approx 1 - 0.9509 \\ &= 0.0491, \end{aligned}$$

or about 4.91%. □

3. Let  $(\mathcal{C}, \mathcal{B}, \Pr)$  denote a probability space, and let  $E_1$ ,  $E_2$  and  $E_3$  be mutually disjoint events in  $\mathcal{B}$ . Find  $\Pr[(E_1 \cup E_2) \cap E_3]$  and  $\Pr(E_1^c \cup E_2^c)$ .

**Solution:** Since  $E_1$ ,  $E_2$  and  $E_3$  are mutually disjoint events, it follows that  $(E_1 \cup E_2) \cap E_3 = \emptyset$ ; so that

$$\Pr[(E_1 \cup E_2) \cap E_3] = 0.$$

Next, use De Morgan's law to compute

$$\begin{aligned} \Pr(E_1^c \cup E_2^c) &= \Pr([E_1 \cap E_2]^c) \\ &= \Pr(\emptyset^c) \\ &= \Pr(\mathcal{C}) \\ &= 1. \end{aligned}$$

□

4. Let  $(\mathcal{C}, \mathcal{B}, \Pr)$  denote a probability space, and let  $A$  and  $B$  events in  $\mathcal{B}$ . Show that

$$\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B). \quad (3)$$

**Solution:** Since  $A \cap B \subseteq A$ , it follows that

$$\Pr(A \cap B) \leq \Pr(A). \quad (4)$$

Similarly, since  $A \subseteq A \cup B$ , we get that

$$\Pr(A) \leq \Pr(A \cup B). \quad (5)$$

Next, use the identity

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$

and fact that that

$$\Pr(A \cap B) \geq 0,$$

to obtain that

$$\Pr(A \cup B) \leq \Pr(A) + \Pr(B). \quad (6)$$

Finally, combine (4), (5) and (6) to obtain (3). □

5. Let  $(\mathcal{C}, \mathcal{B}, \Pr)$  denote a probability space, and let  $E_1$ ,  $E_2$  and  $E_3$  be mutually independent events in  $\mathcal{B}$  with probabilities  $\frac{1}{2}$ ,  $\frac{1}{3}$  and  $\frac{1}{4}$ , respectively. Compute the exact value of  $\Pr(E_1 \cup E_2 \cup E_3)$ .

**Solution:** First, use De Morgan's law to compute

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c \cap E_2^c \cap E_3^c) \quad (7)$$

Then, since  $E_1$ ,  $E_2$  and  $E_3$  are mutually independent events, it follows from (7) that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c) \cdot \Pr(E_2^c) \cdot \Pr(E_3^c),$$

so that

$$\begin{aligned} \Pr[(E_1 \cup E_2 \cup E_3)^c] &= (1 - \Pr(E_1))(1 - \Pr(E_2))(1 - \Pr(E_3)) \\ &= \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}, \end{aligned}$$

so that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{1}{4}. \quad (8)$$

It then follows from (8) that

$$\Pr(E_1 \cup E_2 \cup E_3) = 1 - \Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{3}{4}.$$

□

6. Let  $(\mathcal{C}, \mathcal{B}, \Pr)$  denote a probability space, and let  $E_1$ ,  $E_2$  and  $E_3$  be mutually independent events in  $\mathcal{B}$  with  $\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \frac{1}{4}$ . Compute  $\Pr[(E_1^c \cap E_2^c) \cup E_3]$ .

**Solution:** First, use De Morgan's law to compute

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c \cap E_3^c] \quad (9)$$

Next, use the assumption that  $E_1$ ,  $E_2$  and  $E_3$  are mutually independent events to obtain from (9) that

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c] \cdot \Pr[E_3^c], \quad (10)$$

where

$$\Pr[E_3^c] = 1 - \Pr(E_3) = \frac{3}{4}, \quad (11)$$

and

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - \Pr[E_1^c \cap E_2^c] \\ &= 1 - \Pr[E_1^c] \cdot \Pr[E_2^c], \end{aligned} \quad (12)$$

by the independence of  $E_1$  and  $E_2$ .

It follows from the calculations in (13) that

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - (1 - \Pr[E_1])(1 - \Pr[E_2]) \\ &= 1 - \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{4}\right) \\ &= 1 - \frac{3}{4} \cdot \frac{3}{4} \\ &= \frac{7}{16} \end{aligned} \quad (13)$$

Substitute (11) and the result of the calculations in (13) into (10) to obtain

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \frac{7}{16} \cdot \frac{3}{4} = \frac{21}{64}. \quad (14)$$

Finally, use the result in (14) to compute

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c) \cup E_3^c] &= 1 - \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \\ &= 1 - \frac{21}{64} \\ &= \frac{43}{64}. \end{aligned}$$

□

7. A bowl contains 10 chips of the same size and shape. One and only one of these chips is red. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn. Let  $X$  denote the number of draws needed to get the red chip.

(a) Find the pmf of  $X$ .

**Solution:** Compute

$$\Pr(X = 1) = \frac{1}{10}$$

$$\Pr(X = 2) = \frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10}$$

$$\Pr(X = 3) = \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10}$$

$\vdots$

$$\Pr(X = 10) = \frac{1}{10}$$

Thus,

$$p_x(k) = \begin{cases} \frac{1}{10} & \text{for } k = 1, 2, \dots, 10; \\ 0 & \text{elsewhere.} \end{cases} \quad (15)$$

□

(b) Compute  $\Pr(X \leq 4)$ .

**Solution:** Use (15) to compute

$$\Pr(X \leq 4) = \sum_{k=1}^4 p_x(k) = \frac{4}{10} = \frac{2}{5}.$$

□

8. Let  $X$  have pmf given by  $p_x(x) = \frac{1}{3}$  for  $x = 1, 2, 3$  and  $p(x) = 0$  elsewhere. Give the pmf of  $Y = 2X + 1$ .

**Solution:** Note that the possible values for  $Y$  are 3, 5 and 7  
Compute

$$\Pr(Y = 3) = \Pr(2X + 1 = 3) = \Pr(X = 1) = \frac{1}{3}.$$

Similarly, we get that

$$\Pr(Y = 5) = \Pr(X = 2) = \frac{1}{3},$$

and

$$\Pr(Y = 7) = \Pr(X = 3) = \frac{1}{3}.$$

Thus,

$$p_Y(k) = \begin{cases} \frac{1}{3} & \text{for } k = 3, 5, 7; \\ 0 & \text{elsewhere.} \end{cases}$$

□

9. Let  $X$  have pmf given by  $p_X(x) = \left(\frac{1}{2}\right)^x$  for  $x = 1, 2, 3, \dots$  and  $p_X(x) = 0$  elsewhere. Give the pmf of  $Y = X^3$ .

**Solution:** Compute, for  $y = k^3$ , for  $k = 1, 2, 3, \dots$ ,

$$\Pr(Y = y) = \Pr(X^3 = k^3) = \Pr(X = k) = \left(\frac{1}{2}\right)^k,$$

so that

$$\Pr(Y = y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \quad \text{for } y = k^3, \text{ for some } k = 1, 2, 3, \dots$$

Thus,

$$p_Y(y) = \begin{cases} \left(\frac{1}{2}\right)^{y^{1/3}}, & \text{for } y = k^3, \text{ for some } k = 1, 2, 3, \dots; \\ 0 & \text{elsewhere.} \end{cases}$$

□

10. Let  $f_x(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 < x < \infty; \\ 0 & \text{if } x \leq 1, \end{cases}$  be the pdf of a random variable  $X$ . If  $E_1$  denote the interval  $(1, 2)$  and  $E_2$  the interval  $(4, 5)$ , compute  $\Pr(E_1)$ ,  $\Pr(E_2)$ ,  $\Pr(E_1 \cup E_2)$  and  $\Pr(E_1 \cap E_2)$ .

**Solution:** Compute

$$\Pr(E_1) = \int_1^2 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^2 = \frac{1}{2},$$

$$\Pr(E_2) = \int_4^5 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_4^5 = \frac{1}{20},$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20},$$

since  $E_1$  and  $E_2$  are mutually exclusive, and

$$\Pr(E_1 \cap E_2) = 0,$$

since  $E_1$  and  $E_2$  are mutually exclusive.  $\square$

11. A *mode* of a distribution of a random variable  $X$  is a value of  $x$  that maximizes the pdf or the pmf. If there is only one such value, it is called *the mode of the distribution*. Find the mode for each of the following distributions:

(a)  $p(x) = \left(\frac{1}{2}\right)^x$  for  $x = 1, 2, 3, \dots$ , and  $p(x) = 0$  elsewhere.

**Solution:** Note that  $p(x)$  is decreasing; so,  $p(x)$  is maximized when  $x = 1$ . Thus, 1 is the mode of the distribution of  $X$ .  $\square$

(b)  $f(x) = \begin{cases} 12x^2(1-x), & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere.} \end{cases}$

**Solution:** Maximize the function  $f$  over  $[0, 1]$ .  
Compute

$$f'(x) = 24x(1-x) - 12x^2 = 12x(2-3x),$$

so that  $f$  has a critical points at  $x = 0$  and  $x = \frac{2}{3}$ .

Since  $f(0) = f(1) = 0$  and  $f(2/3) > 0$ , it follows that  $f$  takes on its maximum value on  $[0, 1]$  at  $x = \frac{2}{3}$ . Thus, the mode of the distribution of  $X$  is  $x = \frac{2}{3}$ .  $\square$

12. Let  $X$  have pdf  $f_x(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$

Compute the probability that  $X$  is at least  $3/4$ , given that  $X$  is at least  $1/2$ .

**Solution:** We are asked to compute

$$\Pr(X \geq 3/4 \mid X \geq 1/2) = \frac{\Pr[(X \geq 3/4) \cap (X \geq 1/2)]}{\Pr(X \geq 1/2)}, \quad (16)$$

where

$$\begin{aligned} \Pr(X \geq 1/2) &= \int_{1/2}^1 2x \, dx \\ &= x^2 \Big|_{1/2}^1 \\ &= 1 - \frac{1}{4}, \end{aligned}$$

so that

$$\Pr(X \geq 1/2) = \frac{3}{4}; \quad (17)$$

and

$$\begin{aligned} \Pr[(X \geq 3/4) \cap (X \geq 1/2)] &= \Pr(X \geq 3/4) \\ &= \int_{3/4}^1 2x \, dx \\ &= x^2 \Big|_{3/4}^1 \\ &= 1 - \frac{9}{16}, \end{aligned}$$

so that

$$\Pr[(X \geq 3/4) \cap (X \geq 1/2)] = \frac{7}{16}. \quad (18)$$

Substituting (18) and (17) into (16) then yields

$$\Pr(X \geq 3/4 \mid X \geq 1/2) = \frac{\frac{7}{16}}{\frac{1}{4}} = \frac{7}{12}.$$

□

13. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

**Solution:** Assume the segment is the interval  $(0, 1)$  and let  $X \sim \text{Uniform}(0, 1)$ . Then  $X$  models a random point in  $(0, 1)$ . We have two possibilities: Either  $X \leq 1 - X$  or  $X > 1 - X$ ; or, equivalently,  $X \leq \frac{1}{2}$  or  $X > \frac{1}{2}$ .

Define the events

$$E_1 = \left( X \leq \frac{1}{2} \right) \quad \text{and} \quad E_2 = \left( X > \frac{1}{2} \right).$$

Observe that  $\Pr(E_1) = \frac{1}{2}$  and  $\Pr(E_2) = \frac{1}{2}$ .

The probability that the largest segment is at least three times the shorter is given by

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2),$$

by the Law of Total Probability, where

$$\Pr(1 - X > 3X \mid E_1) = \frac{\Pr[(X < 1/4) \cap E_1]}{\Pr(E_1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Similarly,

$$\Pr(X > 3(1 - X) \mid E_2) = \frac{\Pr[(X > 3/4) \cap E_2]}{\Pr(E_2)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Thus, the probability that the largest segment is at least three times the shorter is

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2) = \frac{1}{2}.$$

□

14. Let  $X$  have pdf  $f_X(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$

Find the pdf of  $Y = X^3$ .

**Solution:** First, compute the cdf of  $Y$

$$F_Y(y) = \Pr(Y \leq y). \quad (19)$$

Observe that, since  $Y = X^3$  and the possible values of  $X$  range from 0 to 3, the values of  $Y$  will range from 0 to 27. Thus, in the calculations that follow, we will assume that  $0 < y < 27$ .

From (19) we get that

$$\begin{aligned} F_Y(y) &= \Pr(X^3 \leq y) \\ &= \Pr(X \leq y^{1/3}) \\ &= F_X(y^{1/3}) \end{aligned}$$

Thus, for  $0 < y < 27$ , we have that

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{1}{3}y^{-3/2}, \quad (20)$$

where we have applied the Chain Rule. It follows from (20) and the definition of  $f_X$  that

$$f_Y(y) = \frac{1}{9} [y^{1/3}]^2 \cdot \frac{1}{3}y^{-3/2} = \frac{1}{27}, \quad \text{for } 0 < y < 27. \quad (21)$$

Combining (21) and the definition of  $f_X$  we obtain the pdf for  $Y$ :

$$f_Y(y) = \begin{cases} \frac{1}{27}, & \text{for } 0 < y < 27; \\ 0 & \text{elsewhere;} \end{cases}$$

in other words  $Y \sim \text{Uniform}(0, 27)$ . □