Solutions to Exam 2 (Part I)

1. Let X denote a discrete random variable with possible values $x_1, x_2, \ldots, x_n$.

   (a) Let g denote a real valued function of a single variable. Give a formula for computing the expectation $E[g(X)]$.

   Answer: $E[g(X)] = \sum_{k=1}^{n} g(x_k)p_X(x_k)$, where p_X is the pmf of X.

   □

   (b) An insurance policy pays \$100 per day for up to 3 days of hospitalization and \$50 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability mass function

$$p_X(k) = \begin{cases} \dfrac{6-k}{15}, & \text{for } k = 1,2,3,4,5; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

   Determine the expected payment for hospitalization under this policy.

   Solution: The payments for hospitalization are described by the function

$$g(x) = \begin{cases} 100, & \text{if } x = 1; \\ 200, & \text{if } x = 2; \\ 300, & \text{if } x = 3; \\ 350, & \text{if } x = 4; \\ 400, & \text{if } x = 5. \end{cases}$$

   where x denotes the number of days of hospitalization. We then have that the expected payment for hospitalization is

$$\begin{aligned} E[g(X)] &= \sum_{k=1}^{5} g(k)p_X(k) \\ &= 100 \cdot \frac{5}{15} + 200 \cdot \frac{4}{15} + 300 \cdot \frac{3}{15} + 350 \cdot \frac{2}{15} + 400 \cdot \frac{1}{15}; \end{aligned}$$

   so that

$$E[g(X)] = 220.$$

   Thus, the expected payment per hospitalization is \$220. □

2. Let X and Y denote independent random variables such that $X \sim \text{Normal}(\mu, 1)$ and $Y \sim \text{Normal}(\mu, 1)$, for some real parameter μ. Define $W = X - Y$.

   (a) Compute the moment generating function of W and use it to determine the distribution of W.

   Solution: Compute

$$\begin{aligned} \psi_W(t) &= \psi_{X-Y}(t) \\ &= E(e^{t(X-Y)}) \\ &= E(e^{tX+(-t)Y}) \\ &= E(e^{tX} \cdot e^{-tY}); \end{aligned}$$

   thus, since X and Y are independent,

$$\begin{aligned} \psi_W(t) &= E(e^{tX}) \cdot E(e^{-tY}) \\ &= \psi_X(t) \cdot \psi_Y(-t). \end{aligned}$$

   Consequently, since $X \sim \text{Normal}(\mu, 1)$ and $Y \sim \text{Normal}(\mu, 1)$,

$$\begin{aligned} \psi_W(t) &= e^{\mu t + \frac{1}{2}t^2} \cdot e^{\mu(-t)+\frac{1}{2}(-t)^2} \\ &= e^{\mu t + \frac{1}{2}t^2} \cdot e^{-\mu t + \frac{1}{2}t^2} \\ &= e^{t^2}, \end{aligned}$$

   which is the mgf of a Normal(0, 2) distribution. It then follows by the Uniqueness Theorem for Moment Generating Functions that

$$W \sim \text{Normal}(0, 2). \tag{1}$$

   □

   (b) Estimate the probability $\Pr(|X - Y| < \sqrt{2})$. Explain the reasoning leading to your answer.

   Solution: It follows from (1) that $\dfrac{X - Y}{\sqrt{2}} \sim \text{Normal}(0, 1)$. Set $Z \dfrac{X - Y}{\sqrt{2}}$;

then, $Z \sim \text{Normal}(0,1)$ and

$$\begin{aligned}
\Pr(|X-Y|<\sqrt{2}) &= \Pr\left(\left|\frac{X-Y}{\sqrt{2}}\right|<1\right) \\
&= \Pr(|Z|<1) \\
&= \Pr(-1<Z<1) \\
&= \Pr(-1<Z\leqslant 1),
\end{aligned}$$

since Z is a continuous random variable. It then follows that

$$\begin{aligned}
\Pr(|X-Y|<\sqrt{2}) &= F_{Z}(1)-F_{Z}(-1) \\
&= F_{Z}(1)-(1-F_{Z}(1)),
\end{aligned}$$

by the symmetry of the pdf of $Z \sim \text{Normal}(0,1)$. We then have that

$$\Pr(|X-Y|<\sqrt{2}) = 2F_{Z}(1)-1,$$

where $F_{Z}(1) \approx 0.8413$. We then have that

$$\Pr(|X-Y|<\sqrt{2}) \approx 0.6826.$$

□