Solutions to Exam 2 (Part II)

1. The random pair (X,Y) has the joint distribution shown in Table 1.

$X\backslash Y$	2	3	4
1	0.1	0	0.1
2	0.3	0.1	0.2
3	0	0.2	0

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

(a) Compute $\Pr(X<Y)$.

Solution:

$$\begin{aligned}\Pr(X<Y) &= p_{(X,Y)}(1,2)+p_{(X,Y)}(1,3)+p_{(X,Y)}(1,4)\\ &\quad p_{(X,Y)}(2,3)+p_{(X,Y)}(2,4)+p_{(X,Y)}(3,4)\\ &= 0.1+0+0.1+0.1+0.2+0\\ &= 0.5.\end{aligned}$$

□

(b) Compute the marginal distributions of X and Y.

Solution: The marginal distribution of X is

$$p_X(k)=\begin{cases}0.2, & \text{if } k=1;\\ 0.6, & \text{if } k=2;\\ 0.2, & \text{if } k=3;\\ 0, & \text{elsewhere};\end{cases}$$

and the marginal distribution of Y is

$$p_Y(k)=\begin{cases}0.4, & \text{if } k=2;\\ 0.3, & \text{if } k=3;\\ 0.3, & \text{if } k=4;\\ 0, & \text{elsewhere};\end{cases}$$

□

(c) Show that X and Y are not independent. Give a reason for your answer.

Solution: Note that $p_{(X,Y)}(1,3) = 0$, while $p_X(1) \cdot p_Y(3) = (0.2) \cdot (0.3)$; so that, $p_X(1) \cdot p_Y(3) = 0.06$. Hence,

$$p_{(X,Y)}(1,3) \neq p_X(1) \cdot p_Y(3),$$

and, therefore, X and Y are not independent. □

(d) Compute the expectations $E(X)$, $E(Y)$ and $E(XY)$.

Solution: Compute

$$E(X) = 1 \cdot (0.2) + 2 \cdot (0.6) + 3 \cdot (0.2) = 2.0;$$

$$E(Y) = 2 \cdot (0.4) + 3 \cdot (0.3) + 4 \cdot (0.3) = 2.9;$$

and

$$\begin{aligned} E(XY) &= 1 \cdot 2(0.1) + 1 \cdot 3(0) + 1 \cdot 4(0.1) \\ &\quad +2 \cdot 2(0.3) + 2 \cdot 3(0.1) + 2 \cdot 4(0.2) \\ &\quad\quad +3 \cdot 2(0) + 3 \cdot 3(0.2) + 3 \cdot 4(0) \\ &= 2(0.1) + 0 + 4(0.1) \\ &\quad +4(0.3) + 6(0.1) + 8(0.2) \\ &\quad\quad +0 + 9(0.2) + 0 \\ &= 0.2 + 0.4 + 1.2 + 0.6 + 1.8 + 1.6 \\ &= 5.8. \end{aligned}$$

□

(e) Compute the covariance of X and Y.

Solution: Compute

$$\text{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y) = 5.8 - (2.0)(2.9) = 0.$$

□

2. A random point (X,Y) is distributed uniformly on the triangle with vertices $(0,0)$, $(4,0)$ and $(0,1)$ in the xy–plane.

   (a) Give a formula for computing the joint pdf, $f_{(X,Y)}$, of the random vector (X,Y).

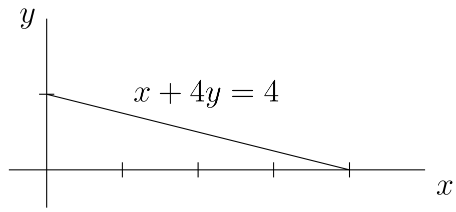

Figure 1: Triangular Region for Problem 2

***Solution**:* Figure 1 shows a sketch of the triangular region with vertices $(0,0)$, $(4,0)$ and $(0,1)$ in the xy–plane. Since the area of the triangle is

$$A = \frac{1}{2}(4)(1) = 2,$$

it follows that the joint pdf of (X, Y) is

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{2}, & \text{if } 0 < x < 4 \text{ and } 0 < y < 1 - x/4; \\ 0, & \text{elsewhere.} \end{cases}$$

□

(b) Compute the marginal distributions f_X and f_Y.
***Solution**:* Compute, for $0 < x < 4$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\, dy \\ &= \int_0^{1-x/4} \frac{1}{2}\, dy \\ &= \frac{1}{2}\left(1 - \frac{x}{4}\right). \end{aligned}$$

Hence,

$$f_X(x) = \begin{cases} \frac{1}{8}(4 - x), & \text{if } 0 < x < 4; \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly, for $0 < y < 1$, compute

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\, dx \\ &= \int_0^{4-4y} \frac{1}{2}\, dy \\ &= \frac{1}{2}(4-4y); \end{aligned}$$

so that

$$f_Y(y) = \begin{cases} 2(1-y), & \text{if } 0 < y < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

□

(c) Are X and Y independent random variables? Give a reason for your answer.

***Solution**:* Observe that

$$f_X(x)\cdot f_Y(y) = \begin{cases} \frac{1}{4}(4-x)(1-y), & \text{if } 0 < x < 4 \text{ and } 0 < y < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, $f_{(X,Y)}(x,y) \neq f_X(x)\cdot f_Y(y)$ for all $(x,y)\in\mathbb{R}^2$. Hence, X and Y are not independent. □

(d) Compute the expectations $E(X)$, $E(Y)$ and $E(XY)$

***Solution**:* Compute

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^4 \frac{x}{8}(4-x)\, dx \\ &= \frac{1}{8}\int_0^4 (4x - x^2)\, dx \\ &= \frac{1}{8}\left[2x^2 - \frac{x^3}{3}\right]_0^4; \end{aligned}$$

so that

$$E(X) = \frac{4}{3}.$$

Similarly,

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} x f_{_Y}(y)\ dy \\ &= \int_0^1 2y(1-y)\ dy \\ &= \int_0^1 (2y - 2y^2)\ dy \\ &= \left[y^2 - \frac{2}{3}y^3 \right]_0^1; \end{aligned}$$

so that

$$E(Y) = \frac{1}{3}.$$

Next, compute

$$\begin{aligned} E(XY) &= \iint_{\mathbb{R}^2} xy f_{_{(X,Y)}}(x,y)\ dxdy \\ &= \frac{1}{2} \int_0^1 \int_0^{4(1-y)} xy\ dxdy \\ &= \frac{1}{4} \int_0^1 \left[x^2 y \right]_0^{4(1-y)}\ dy \\ &= 4 \int_0^1 (1-y)^2 y\ dy \\ &= 4 \int_0^1 (y - 2y^2 + y^3)\ dy \\ &= 4 \left[\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4} \right]_0^1; \end{aligned}$$

so that

$$E(XY) = \frac{1}{3}.$$

□

(e) Compute the covariance of X and Y.
Solution: Compute

$$\begin{aligned}\text{Cov}(X,Y) &= E(XY) - E(X)\cdot E(Y)\\ &= \frac{1}{3} - \frac{4}{3}\cdot\frac{1}{3};\end{aligned}$$

so that,

$$\text{Cov}(X,Y) = -\frac{1}{9}.$$

□

3. Suppose that the joint pdf of the random vector (X,Y) is given by

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}, \quad \text{for } -\infty < x < \infty \text{ and } -\infty < y < \infty.$$

(a) Verify that $f_{(X,Y)}$ is indeed a joint pdf.
Solution: Use polar coordinates to evaluate

$$\begin{aligned}\iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\, dxdy &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dxdy\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, rdrd\theta\\ &= \int_0^{\infty} e^{-r^2/2}\, rdr;\end{aligned}$$

thus, making the change of variables $u = r^2/2$,

$$\iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\, dxdy = \int_0^{\infty} e^{-u}\, du = 1.$$

Hence, $f_{(X,Y)}$ is indeed a joint pdf. □

Alternate Solution: Observe that

$$f_{(X,Y)}(x,y) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\cdot\frac{1}{\sqrt{2\pi}}e^{-y^2/2}, \tag{1}$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, which can be written as

$$f_{(X,Y)}(x,y) = f_X(x)\cdot f_Y(y), \quad \text{for } -\infty < x < \infty \text{ and } -\infty < y < \infty, \tag{2}$$

where f_X and f_Y are both pdfs of a standard normal random variable. It then follows that

$$\iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\ dxdy = \int_{-\infty}^{\infty} f_X(x)\ dx \int_{-\infty}^{\infty} f_Y(y)\ dy = 1.$$

□

(b) Compute the marginal distributions f_X and f_Y.

***Solution**:* It follows from (1) that

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad \text{for } -\infty < x < \infty, \tag{3}$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}, \quad \text{for } -\infty < y < \infty. \tag{4}$$

□

(c) Are X and and Y independent? Give a reason for your answer.

***Solution**:* Yes; this follows from (2). □

(d) Determine the distribution of $X+Y$.

***Solution**:* It follows from (3) and (4) that $X \sim \text{Normal}(0,1)$ and $Y \sim \text{Normal}(0,1)$; thus, since X and Y are independent, as seen in the previous part,

$$X + Y \sim \text{Normal}(0,2). \tag{5}$$

□

(e) Compute $\Pr(-\sqrt{2} < X+Y \leqslant 2\sqrt{2})$.

***Solution**:* Set $Z = \dfrac{X+Y}{\sqrt{2}}$. It then follows from (5) that $Z \sim \text{Normal}(0,1)$. Then,

$$\begin{aligned}\Pr(-\sqrt{2} < X+Y \leqslant 2\sqrt{2}) &= \Pr\left(-1 < \frac{X+Y}{\sqrt{2}} \leqslant 2\right) \\ &= \Pr\left(-1 < Z \leqslant 2\right),\end{aligned}$$

where $Z \sim \text{Normal}(0,1)$. Consequently,

$$\begin{aligned}
\Pr(-\sqrt{2} < X+Y \leqslant 2\sqrt{2}) &= F_Z(2) - F_Z(-1) \\
&= F_Z(2) - (1 - F_Z(1)) \\
&= F_Z(2) + F_Z(1) - 1.
\end{aligned}$$

We then have that

$$\Pr(-\sqrt{2} < X+Y \leqslant 2\sqrt{2}) \approx 0.9772 + 0.8413 - 1 = 0.8185.$$

□