Solutions to Exam 3 (Part I)

1. Let $X_1, X_2, X_3 \ldots$ denote a sequence of random variables.

   (a) State the Central Limit Theorem in the context of the sequence (X_k).

   Answer: Let (X_k) be a sequence of independent, identically distributed random variables of mean μ and variance σ^2. Then,

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} Z \sim \text{Normal}(0,1) \text{ as } n \to \infty.$$

□

   (b) Let $X_1, X_2, \ldots, X_n$ denote random sample of size 49 from a Uniform$(0,1)$ distribution. Let $\overline{X}_n$ denote the sample mean. Use the Central Limit Theorem to estimate the probability

$$\Pr(0.4 < \overline{X}_n < 0.6).$$

   Solution: In this case, the random variables, X_k, are independent, identically distributed with mean $\mu = \dfrac{1}{2}$ and variance $\sigma^2 = \dfrac{1}{12}$; or $\mu = 0.5$ and $\sigma^2 \doteq 0.083$. Thus, applying the Central Limit Theorem,

$$\Pr(0.4 < \overline{X}_n < 0.6) \approx \Pr\left(\frac{0.4 - 0.5}{0.289/\sqrt{49}} < Z < \frac{0.6 - 0.5}{0.289/\sqrt{49}}\right),$$

   where $Z \sim \text{Normal}(0,1)$, or

$$\begin{aligned}\Pr(0.4 < \overline{X}_n < 0.6) &\approx \Pr(-2.42 < Z < 2.42) \\ &= \Pr(-2.42 < Z \leqslant 2.42) \\ &= F_Z(2.42) - F_Z(-2.42);\end{aligned}$$

   so that,

$$\Pr(0.4 < \overline{X}_n < 0.6) \approx 2F_Z(2.42) - 1, \qquad (1)$$

   where $Z \sim \text{Normal}(0,1)$.
   Using a table of standard normal probabilities we obtain from (1) that

$$\Pr(0.4 < \overline{X}_n < 0.6) \approx 2(0.9922) - 1,$$

   or

$$\Pr(0.4 < \overline{X}_n < 0.6) \approx 0.9844,$$

   or about 98.44%. □

2. Let X denote a random variable with mean μ and variance σ^2.

   (a) State the Chebyshev Inequality.

   Answer: Let X be a random variable with mean μ and finite variance $\text{Var}(X)$; then, for every $\varepsilon > 0$,

$$\Pr(|X - \mu| \geqslant \varepsilon) \leqslant \frac{\text{Var}(X)}{\varepsilon^2}. \tag{2}$$

□

   (b) Let $X_1, X_2, \ldots, X_n$ denote random sample of size n from a distribution of mean μ and variance σ^2. Apply the Chebyshev inequality to get an upper bound for the probability

$$\Pr(|\overline{X}_n - \mu| \geqslant k\sigma),$$

   where k is a positive number.

   Solution: Applying (2) to $X = \overline{X}_n$ and $\varepsilon = k\sigma$, so that

$$E(\overline{X}_n) = \mu \quad \text{and} \quad \text{Var}(\overline{X}_n) = \frac{\sigma^2}{n},$$

   we obtain that

$$\Pr(|\overline{X}_n - \mu| \geqslant k\sigma) \leqslant \frac{\sigma^2/n}{(k\sigma)^2};$$

   so that

$$\Pr(|\overline{X}_n - \mu| \geqslant k\sigma) \leqslant \frac{1}{nk^2}. \tag{3}$$

□

   (c) Let Y_n denote the number of heads in n tosses of a fair coin. Use the result from part (b) to obtain an upper bound for the the probability that Y_n deviates from $n/2$ by more than $5\sqrt{n}$.

   Solution: Let (X_k) denote a sequence of independent Bernoulli(1/2) trials. We then have that

$$Y_n = \sum_{k=1}^{n} X_k;$$

   so that

$$Y_n = n\overline{X}_n. \tag{4}$$

   Use (4) to compute

$$\Pr(|Y_n - n/2| \geqslant 5\sqrt{n}) = \Pr(|\overline{X}_n - 1/2| \geqslant 5/\sqrt{n}) \tag{5}$$

Next, apply the result in (3) with

$$\mu = E(X_k) = \frac{1}{2}, \quad \sigma^2 = \text{Var}(X_k) = \frac{1}{4}, \quad \text{and} \quad k = \frac{10}{\sqrt{n}}$$

to obtain from (5) that

$$\Pr(|Y_n - n/2| \geqslant 5\sqrt{n}) = \Pr\left(|\overline{X}_n - 1/2| \geqslant \frac{10}{\sqrt{n}} \cdot \frac{1}{2}\right) \leqslant \frac{1}{n(10/\sqrt{n})^2};$$

so that

$$\Pr(|Y_n - n/2| \geqslant 5\sqrt{n}) \leqslant \frac{1}{100}.$$

□