Review Problems for Exam 3

1. Let X denote a positive random variable such that $\ln(X)$ has a Normal$(0, 1)$ distribution.

   (a) Give the pdf of X and compute its expectation.

   (b) Estimate $\Pr(X \leq 6.5)$.

2. Forty seven digits are chosen at random and with replacement from $\{0, 1, 2, \ldots, 9\}$. Estimate the probability that their average lies between 4 and 6.

3. Let $X_1, X_2, \ldots, X_{30}$ be independent random variables each having a discrete distribution with pmf:

$$p(x) = \begin{cases} 1/4, & \text{if } x = 0 \text{ or } x = 2; \\ 1/2 & \text{if } x = 1; \\ 0 & \text{otherwise.} \end{cases}$$

   Estimate the probability that $X_1 + X_2 + \cdots + X_{30}$ is at most 33.

4. Roll a balanced die 36 times. Let Y denote the sum of the outcomes in each of the 36 rolls. Estimate the probability that $108 \leq Y \leq 144$.

   Suggestion: Since the event of interest is $(Y \in \{108, 109, \ldots, 144\})$, rewrite $\Pr(108 \leq Y \leq 144)$ as

$$\Pr(107.5 < Y \leq 144.5).$$

5. Let $Y \sim$ Binomial$(100, 1/2)$. Use the Central Limit Theorem to estimate the value of $\Pr(Y = 50)$.

   Suggestion: Observe that

$$\Pr(Y = 50) = \Pr(49.5 < Y \leqslant 50.5),$$

   since Y is discrete.

6. Let $Y \sim$ Binomial$(n, 0.55)$. Find the smallest value of n such that, approximately,

$$\Pr(Y/n > 1/2) \geq 0.95.$$

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ. Thus, $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\lambda$. Moreover, by the Central Limit Theorem, $\overline{X} = Y/n$ has, approximately, a Normal$(\lambda, \lambda/n)$ distribution for large n. Show that $u(Y/n) = \sqrt{Y/n}$ is a function of Y/n which is essentially free of λ.

8. Suppose that factory produces a number X of items in a week, where X can be modeled by a random variable with mean 50. Suppose also that the variance for a week's production is known to be 25. What can be said about the probability that this week's production will be between 40 and 60?

9. Let (X_n) denote a sequence of nonnegative random variables with means $\mu_n = E(X_n)$, for each $n = 1, 2, 3, \ldots$. Assume that $\lim_{n\to\infty} \mu_n = 0$. Show that X_n converges in probability to 0 as $n \to \infty$.

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ. Thus, $Y_n = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\lambda$. Moreover, by the Central Limit Theorem, $\overline{X}_n = Y_n/n$ has, approximately, a Normal$(\lambda, \lambda/n)$ distribution for large n. Show that, for large values of n, the distribution of $2\sqrt{n}\left(\sqrt{\frac{Y_n}{n}} - \sqrt{\lambda}\right)$ is independent of λ.