Solutions to Review Problems for Final Exam

1. Three cards are in a bag. One card is red on both sides. Another card is white on both sides. The third card in red on one side and white on the other side. A card is picked at random and placed on a table. Compute the probability that if a given color is shown on top, the color on the other side is the same as that of the top.

   ***Solution**:* Each card has a likelihood of 1/3 of being picked.

   Assume for definiteness that the top of the picked card is red. Let T_r denote the event that the top of the picked car shows red and B_r denote the event that the bottom of the card is also red. We want to compute

   $$\Pr(B_r \mid T_r) = \frac{\Pr(T_r \cap B_r)}{\Pr(T_r)}. \tag{1}$$

   Note that

   $$\Pr(T_r \cap B_r) = \frac{1}{3}, \tag{2}$$

   since there is only one card for which both sides are red.

   In order to compute $\Pr(T_r)$ observe that there are three equally likely choices out of six for the top of the card to show red; thus,

   $$\Pr(T_r) = \frac{1}{2}. \tag{3}$$

   Hence, using (2) and (3), we obtain from (1) that

   $$\Pr(B_r \mid T_r) = \frac{2}{3}. \tag{4}$$

   Similar calculations can be used to show that

   $$\Pr(T_w) = \frac{1}{2}, \tag{5}$$

   and

   $$\Pr(B_w \mid T_w) = \frac{2}{3}. \tag{6}$$

   Let E denote the event that a card showing a given color on the top side will have the same color on the bottom side. Then, by the law of total probability,

   $$\Pr(E) = \Pr(T_r) \cdot \Pr(B_r \mid T_r) + \Pr(T_w) \cdot \Pr(B_w \mid T_w), \tag{7}$$

so that, using (2), (4), (5) and (6), we obtain from (7) that

$$\Pr(E) = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}.$$

$\square$

2. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and a number b of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Determine the value of b.

   ***Solution**:* Let R_1 denote the event that the ball drawn from the first urn is red, B_1 denote the event that the ball drawn from the first urn is blue, R_2 denote the event that the ball drawn from the second urn is red, and B_2 denote the event that the ball drawn from the second urn is blue. We are interested in the event

   $$E = (R_1 \cap R_2) \cup (B_1 \cap B_2),$$

   the event that both balls are of the same color. We observe that $R_1 \cap R_2$ and $B_1 \cap B_2$ mutually exclusive events; thus,

   $$\Pr(E) = \Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2), \tag{8}$$

   where R_1 and R_2 are independent events as well as B_1 and B_2. It the follows from (8) that

   $$\Pr(E) = \Pr(R_1) \cdot \Pr(R_2) + \Pr(B_1) \cdot \Pr(B_2), \tag{9}$$

   where

   $$\Pr(R_1) = \frac{4}{10}, \quad \Pr(R_2) = \frac{16}{16+b}, \quad \Pr(B_1) = \frac{6}{10}, \quad \text{and} \quad \Pr(B_2) = \frac{b}{16+b};$$

   thus, substituting into (9),

   $$\Pr(E) = \frac{4}{10} \cdot \frac{16}{16+b} + \frac{6}{10} \cdot \frac{b}{16+b}. \tag{10}$$

   We are given that $\Pr(E) = 0.44$; combining this information with (10) yields

   $$\frac{32+3b}{16+b} = 2.2. \tag{11}$$

   Solving (11) for b yields $b = 4$. $\square$

3. A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

   ***Solution**:* Let D denote the event that a person selected at random from the population has the disease. Then,

   $$\Pr(D) = 0.01. \tag{12}$$

   Let P denote the event that the blood test is positive for the existence of the disease. We are given that

   $$\Pr(P \mid D) = 0.95 \tag{13}$$

   and

   $$\Pr(P \mid D^c) = 0.005. \tag{14}$$

   We want to compute

   $$\Pr(D \mid P) = \frac{\Pr(D \cap P)}{\Pr(P)}, \tag{15}$$

   where

   $$\Pr(D \cap P) = \Pr(D) \cdot \Pr(P \mid D),$$

   by the Multiplication Rule; so that,

   $$\Pr(D \cap P) = 0.0095, \tag{16}$$

   by virtue of (12) and (13), and

   $$\Pr(P) = \Pr(D) \cdot \Pr(P \mid D) + \Pr(D^c) \cdot \Pr(P \mid D^c), \tag{17}$$

   by the Law of Total Probability.

   Substituting the values in (12), (13) and (14) into (17) yields

   $$\Pr(P) = (0.01) \cdot (0.95) + (0.99) \cdot (0.005) = 0.01445. \tag{18}$$

   Substituting the values in (16) and (18) into (15) yields

   $$\Pr(D \mid P) \doteq 0.6574.$$

   Thus, it a person tests positive, there is about a 66% chance that she or he has the disease. $\square$

4. A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

   ***Solution**:* Let X denote the number of participants in the first group that drop out of the study and Y denote the number of participants in the second group that drop out of the study. Then X and Y are independent Binomial$(10, 0.2)$ random variables. Then, $A = (X \leqslant 1)$ is the event that at least 9 participants complete the study in the first group, and $B = (Y \leqslant 1)$ is the event that at least 9 participants complete the study in the second group. We cant to compute the probability of the event

   $$E = (A \cap B^c) \cup (A^c \cap B),$$

   so that

   $$\Pr(E) = \Pr(A \cap B^c) + \Pr(A^c \cap B), \tag{19}$$

   since $A \cap B^c$ and $A^c \cap B$ are mutually exclusive.

   Next, use the independence of A and B, given that X and Y are independent random variables, to get from (19) that

   $$\Pr(E) = \Pr(A) \cdot (1 - \Pr(B) + \Pr(A) \cdot (1 - \Pr(B)), \tag{20}$$

   where

   $$\begin{aligned} \Pr(A) &= \Pr(X \leqslant 1) \\ &= \Pr(X = 0) + \Pr(X = 1) \\ &= (0.8)^{10} + 10 \cdot (0.2) \cdot (0.8)^9, \end{aligned}$$

   so that

   $$\Pr(A) \doteq 0.3758. \tag{21}$$

   Similarly,

   $$\Pr(B) \doteq 0.3758. \tag{22}$$

   Substituting the results in (21) and (22) into (20) yields

   $$\Pr(E) \doteq 0.4691.$$

   Thus, the probability that at least 9 participants complete the study in one of the two groups, but not in both groups, is about 47%. □

5. Suppose that $0 < \rho < 1$ and let $p(n) = \rho^n(1-\rho)$ for $n = 0, 1, 2, 3, \ldots$

   (a) Verify that p is the probability mass function (pmf) for a random variable.
   Solution: Compute

$$\sum_{n=0}^{\infty} \rho^n(1-\rho) = (1-\rho)\sum_{n=0}^{\infty} \rho^n = (1-\rho)\cdot\frac{1}{1-\rho} = 1,$$

   since $0 < \rho < 1$ and, therefore, the geometric series $\displaystyle\sum_{n=0}^{\infty} \rho^n$ converges to $\dfrac{1}{1-\rho}$. □

   (b) Let X denote a discrete random variable with pmf p. Compute $\Pr(X > 1)$.
   Solution: Compute

$$\begin{aligned} \Pr(X > 1) &= 1 - \Pr(X \leqslant 1) \\ &= 1 - p(0) - p(1) \\ &= 1 - (1-\rho) - \rho(1-\rho) \\ &= \rho^2. \end{aligned}$$

□

6. Let $N(t)$ denote the number of mutations in a bacterial colony that occur during the interval $[0, t)$. Assume that $N(t) \sim \text{Poisson}(\lambda t)$ where $\lambda > 0$ is a positive parameter.

   (a) Give an interpretation for λ.

   Answer: λ is the average number of mutations per unit of time. To see why this assertion makes sense, note that, since $N(t) \sim \text{Poisson}(\lambda t)$,

$$E(N(t)) = \lambda t, \quad \text{for } t > 0.$$

   Thus,

$$\lambda = \frac{E(N(t))}{t}, \quad \text{for } t > 0.$$

□

(b) Let T_1 denote the time that the first mutation occurs. Find the distribution of T_1.

***Solution**:* Observe that, for $t > 0$, the event $[T_1 > t]$ is the same as the event $[N(t) = 0]$; that is, if $t < T_1$, there have have been no mutations in the time interval $[0, t]$. Consequently,

$$\Pr[T_1 > t] = \Pr[N(t) = 0] = e^{-\lambda t},$$

since $N(t) \sim \text{Poisson}(\lambda t)$. Thus,

$$\Pr[T_1 \leqslant t] = 1 - \Pr[T_1 > t] = 1 - e^{-\lambda t}, \qquad \text{for } t > 0.$$

We then have that the cdf of T_1 is

$$F_{T_1}(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0. \end{cases} \tag{23}$$

It follows from (23) that the pdf for T_1 is

$$f_{T_1}(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0, \end{cases}$$

which is the pdf for an exponential distribution with parameter $\beta = 1/\lambda$; thus,

$$T_1 \sim \text{Exponential}(1/\lambda).$$

□

7. Let $X \sim \text{Exponential}(\beta)$, for $\beta > 0$. Compute the median of X.

***Solution**:* In general, a median for the distribution of a random variable, X, is a real value m, for which

$$\Pr(X \leqslant m) \geqslant \frac{1}{2} \quad \text{and} \quad \Pr(X \geqslant m) \geqslant \frac{1}{2}.$$

For the case of a continuous random variable, X, there is only one median, m, obtained from

$$\Pr(X \leqslant m) = \frac{1}{2};$$

in other words, the value m for which

$$F_X(m) = \frac{1}{2}. \tag{24}$$

The cdf for $X \sim \text{Exponential}(\beta)$ is given by

$$F_X(x) = \begin{cases} 1 - e^{-x/\beta}, & \text{for } x > 0; \\ 0 & \text{for } x \leqslant 0. \end{cases} \tag{25}$$

Thus, in view of (24) and (25), we see that the median of the distribution of $X \sim \text{Exponential}(\beta)$ is a positive value, m, such that

$$1 - e^{-m/\beta} = \frac{1}{2}. \tag{26}$$

Solving (26) for m yields

$$m = \beta \ln(2).$$

□

8. Two checkers at a service station complete checkouts independent of one another in times $T_1 \sim \text{Exponential}(\mu_1)$ and $T_2 \sim \text{Exponential}(\mu_2)$, respectively. That is, one checker serves $1/\mu_1$ customers per unit time on average, while the other serves $1/\mu_2$ customers per unit time on average.

   (a) Give the joint pdf, $f_{T_1,T_2}(t_1, t_2)$, of T_1 and T_2.
   ***Solution**:* Since T_1 and T_2 are independent random variables, the joint pdf of (T_1, T_2) is given by

   $$f_{(T_1,T_2)}(x, y) = f_{T_1}(x) \cdot f_{T_1}(y), \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{27}$$

   where

   $$f_{T_1}(x) = \begin{cases} \dfrac{1}{\mu_1}\, e^{-x/\mu_1}, & \text{for } x > 0; \\ \\ 0 & \text{for } x \leqslant 0, \end{cases} \tag{28}$$

   and

   $$f_{T_2}(y) = \begin{cases} \dfrac{1}{\mu_2}\, e^{-y/\mu_2}, & \text{for } y > 0; \\ \\ 0 & \text{for } y \leqslant 0. \end{cases} \tag{29}$$

   It follows from (27), (27) and (29) that the joint pdf of (T_1, T_2) is

   $$f_{(T_1,T_2)}(t_1, t_2) = \begin{cases} \dfrac{1}{\mu_1 \mu_2}\, e^{-t_1/\mu_1 - t_2/\mu_2}, & \text{for } t_1 > 0 \text{ and } t_2 > 0; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

   □

(b) Define the minimum service time, T_m, to be $T_m = \min\{T_1, T_2\}$. Determine the type of distribution that T_m has and give its pdf, $f_{T_m}(t)$.

Solution: Observe that, for $t > 0$, the event $[T_m > t]$ is the same as the event $[T_1 > t, T_2 > t]$, since T_m is the smaller of T_1 and T_2. Consequently,

$$\Pr[T_m > t] = \Pr[T_1 > t, T_2 > t]. \tag{30}$$

Thus, by the independence of T_1 and T_2, it follows from (30) that

$$\Pr[T_m > t] = \Pr[T_1 > t] \cdot \Pr[T_2 > t], \tag{31}$$

where

$$\Pr[T_i > t] = 1 - F_{T_i}(t) = 1 - (1 - e^{-t/\mu_i}),$$

so that

$$\Pr[T_i > t] = e^{-t/\mu_i}, \quad \text{for } t > 0 \text{ and } i = 1, 2. \tag{32}$$

Combining (31) and (32) then yields

$$\Pr[T_m > t] = e^{-t/mu_1 - t/\mu_2},$$

or

$$\Pr[T_m > t] = e^{-t/\beta}, \tag{33}$$

where we have set

$$\frac{1}{\beta} = \frac{1}{\mu_1} + \frac{1}{\mu_2}. \tag{34}$$

It follows from (33) that the cdf of T_m is

$$F_{T_m}(t) = \begin{cases} 1 - e^{-t/\beta}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0, \end{cases}$$

where β is given by (34). Thus, the pdf for T_m is

$$f_{T_m}(t) = \begin{cases} \dfrac{1}{\beta}\, e^{-t/\beta}, & \text{for } t > 0; \\ \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{35}$$

which is the pdf for an exponential distribution with parameter β given by (34); thus,

$$T_m \sim \text{Exponential}(\beta), \quad \text{where } \beta = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}. \tag{36}$$

$\square$

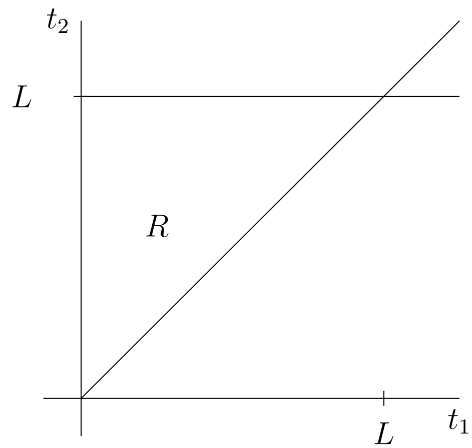

Figure 1: Region for Problem 9

(c) Suppose that, on average, one of the checkers serves 4 customers in an hour, and the other serves 6 customers per hour. On average, what is the minimum amount of time that a customer will spend being served at the service station?

Solution: We compute the expected value of T_m, where T_m has pdf given in (35) with

$$\beta = \frac{\dfrac{1}{4}\cdot\dfrac{1}{6}}{\dfrac{1}{4}+\dfrac{1}{6}} = \frac{1}{10},$$

in view of (36). Thus, on average, the minimum time spent by a customer being served at the service station is one tenth of an hour, or 6 minutes. $\square$

9. Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \leqslant t_1 \leqslant t_2 \leqslant L$, where L is a positive constant. Determine the expected value of the sum of the squares of T_1 and T_2.

Solution: The region $R = \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leqslant t_1 \leqslant t_2 \leqslant L\}$ is pictured in Figure 1. The area of R is

$$\text{area}(R) = \frac{L^2}{2}.$$

Thus, since (T_1, T_2) has a uniform distribution over R, it follows that the joint pdf of T_1 and T_2 is given by

$$f_{(T_1,T_2)}(t_1,t_2) = \begin{cases} \dfrac{2}{L^2}, & \text{for } 0 \leqslant t_1 \leqslant t_2 \leqslant L; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

We want to compute $E(T_1^2 + T_2^2)$, or

$$\begin{aligned} E(T_1^2 + T_2^2) &= \iint_{\mathbb{R}^2} (t_1^2 + t_2^2) f_{(T_1,T_2)}(t_1,t_2)\ dt_1 dt_2 \\ &= \iint_R (t_1^2 + t_2^2) \frac{2}{L^2}\ dt_1 dt_2 \\ &= \frac{2}{L^2} \int_0^L \int_{t_1}^L (t_1^2 + t_2^2)\ dt_2 dt_1 \\ &= \frac{2}{L^2} \int_0^L \left[(t_1^2 t_2 + \frac{t_2^3}{3}) \right]_{t_1}^L\ dt_1 \\ &= \frac{2}{L^2} \int_0^L \left[L t_1^2 + \frac{L^3}{3} - \left(t_1^3 + \frac{t_1^3}{3} \right) \right]\ dt_1 \\ &= \frac{2}{L^2} \int_0^L \left[\frac{L^3}{3} + L t_1^2 - \frac{4}{3} t_1^3 \right]\ dt_1 \\ &= \frac{2}{L^2} \left[\frac{L^3}{3} t_1 + \frac{1}{3} L t_1^3 - \frac{1}{3} t_1^4 \right]_0^L, \end{aligned}$$

so that

$$E(T_1^2 + T_2^2) = \frac{2}{3} L^2.$$

□

10. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \frac{x+y}{8}, \quad \text{for } 0 < x < 2 \text{ and } 0 < y < 2;$$

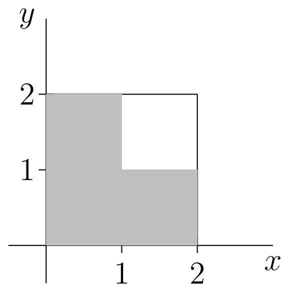

Figure 2: Region for Problem 10

and 0 elsewhere.

What is the probability that the device fails during its first hour of operation?

Solution: We want to compute the probability that either component fails within the first hour of operation; that is the probability of the event E given by

$$E = (0 < X < 1) \cup (0 < Y < 1).$$

The event E is pictured as the shaded region in Figure 2.

The probability of E is given by

$$\begin{aligned}
\Pr(E) &= \iint_E f(x, y)\, dxdy \\
&= \int_0^1 \int_0^2 \frac{x+y}{8}\, dydx + \int_1^2 \int_0^1 \frac{x+y}{8}\, dydx \\
&= \frac{1}{8}\left(\int_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 dx + \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^1 dx\right) \\
&= \frac{1}{8}\left(\int_0^1 (2x+2)\, dx + \int_1^2 \left(x + \frac{1}{2}\right) dx\right) \\
&= \frac{1}{8}\left(\left[x^2 + 2x\right]_0^1 + \left[\frac{x^2}{2} + \frac{x}{2}\right]_1^2\right),
\end{aligned}$$

so that $\Pr(E) = \dfrac{1}{8}(3 + (3-1)) = \dfrac{5}{8}$. Thus, the probability that the device fails during its first hour of operation is 62.5%. $\square$

11. A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E[X]$.

    ***Solution**:* The pdf of T is

$$f_{T}(t) = \begin{cases} \dfrac{1}{3}\, e^{-t/3}, & \text{for } t > 0; \\ \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{37}$$

where t is measured in years. The random variable $X = \max(T, 2)$ is also given by

$$X(T) = \begin{cases} 2, & \text{if } 0 \leqslant T \leqslant 2; \\ T, & \text{if } T > 2. \end{cases}$$

Thus, the expected value of $X(T)$ is given by

$$E(X(T)) = \int_{-\infty}^{\infty} X(t) f_{T}(t)\; dt,$$

where f_{T} is as given in (37) and

$$X(t) = \begin{cases} 2, & \text{if } 0 \leqslant t \leqslant 2; \\ t, & \text{if } t > 2. \end{cases}$$

We then have that

$$\begin{aligned} E(X(T)) &= \int_0^2 2 \cdot \frac{1}{3} e^{-t/3}\; dt + \int_2^\infty t \cdot \frac{1}{3} e^{-t/3}\; dt \\ &= 2\left[-e^{-t/3}\right]_0^2 + \left[-te^{-t/3} - 3e^{-t/3}\right]_2^\infty, \end{aligned}$$

where we have used integration by parts to integrate the last integral. We then have that

$$E(X(T)) = 2\left(1 - e^{-2/3}\right) + 2e^{-2/3} + 3e^{-2/3} = 2 + 3e^{-2/3}.$$

□

12. The number, Y, of spam messages sent to a server in a day has a Poisson distribution with parameter $\lambda = 21$. Each spam message independently has a probability $p = 1/3$ of not being detected by the spam filter. Let X denote the number of spam massages getting through the filter. Calculate the expected daily number of spam messages which get into the server.

Solution: We are given that

$$Y \sim \text{Poisson}(\lambda), \tag{38}$$

where $\lambda = 21$, so that

$$\Pr[Y = n] = \frac{\lambda^n}{n!}\, e^{-\lambda}, \quad \text{for } n = 0, 1, 2, \ldots$$

Now, since each spam message independently has a probability $p = 1/3$ of not being detected by the spam filter, it follows that X has a conditional distribution (conditioned on $Y = n$) that is Binomial(n, p), with $p = 1/3$; thus,

$$\Pr[X = k \mid Y = n] = \begin{cases} \dbinom{n}{k} p^k (1-p)^{n-k}, & \text{for } k = 0, 1, 2, \ldots, n; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{39}$$

Then,

$$\begin{aligned} \Pr[X = k] &= \sum_{n=0}^{\infty} \Pr[X = k, Y = n] \\ &= \sum_{n=0}^{\infty} \Pr[Y = n] \cdot \Pr[X = k \mid Y = n], \end{aligned}$$

where $\Pr[X = k \mid Y = n] = 0$ for $n < k$, so that, using (38) and (39),

$$\begin{aligned} \Pr[X = k] &= \sum_{n=k}^{\infty} \frac{\lambda^n}{n!}\, e^{-\lambda} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{e^{-\lambda}}{k!}\, p^k \sum_{n=k}^{\infty} \lambda^n \frac{1}{(n-k)!} (1-p)^{n-k}. \end{aligned} \tag{40}$$

Next, make the change of variables $\ell = n - k$ in the last summation in (40) to get

$$\Pr[X = k] = \frac{e^{-\lambda}}{k!}\, p^k \sum_{\ell=0}^{\infty} \lambda^{\ell+k} \frac{1}{\ell!} (1-p)^{\ell},$$

so that

$$\begin{aligned}
\Pr[X=k] &= \frac{(\lambda p)^k}{k!}\, e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{1}{\ell!}[\lambda(1-p)]^\ell \\
&= \frac{(\lambda p)^k}{k!}\, e^{-\lambda} e^{\lambda(1-p)} \\
&= \frac{(\lambda p)^k}{k!}\, e^{-\lambda p},
\end{aligned}$$

which shows that $X \sim \text{Poisson}(\lambda p)$, or $X \sim \text{Poisson}(7)$ We then have that $E(X)=7$, so that the expected daily number of spam messages which get into the server is 7. □

13. The lifetime of a printer costing \$200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

    ***Solution**:* Let T denote the lifetime of a printer; then, the pdf of T is

$$f_T(t) = \begin{cases} \dfrac{1}{2}\, e^{-t/2}, & \text{for } t>0; \\ \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{41}$$

    where t is measured in years.

    The expected cost on the sale of one printer is

$$E(C) = 200 \cdot \Pr(0 < T \leqslant 1) + 100 \cdot \Pr(1 < T \leqslant 2), \tag{42}$$

    where, using the pdf in (41),

$$\Pr(0 < T \leqslant 1) = \int_0^1 \frac{1}{2}\, e^{-t/2}\, dt = 1 - e^{-1/2}, \tag{43}$$

    and

$$\Pr(1 < T \leqslant 2) = \int_1^2 \frac{1}{2}\, e^{-t/2}\, dt = \left[-e^{-t/2}\right]_1^2 = e^{-1/2} - e^{-1}. \tag{44}$$

    Substituting the results in (43) and (44) into (42) yields

$$E(C) = 200 - 100e^{-1} - 100e^{-1/2} \doteq 102.56.$$

    Thus, the expected payments because of returns on the sale of 100 printers is about \$10,256. □

14. A computer manufacturing company conducts acceptance sampling for incoming computer chips. After receiving a huge shipment of computer chips, the company randomly selects 800 chips. If three or fewer nonconforming chips are found, the entire lot is accepted without inspecting the remaining chips in the lot. If four or more chips are nonconforming, every chip in the entire lot is carefully inspected at the supplier's expense. Assume that the true proportion of nonconforming computer chips being supplied is 0.001. Estimate the probability the lot will be accepted.

    ***Solution**:* Let X denote the number of nonconforming chips found in the random sample of 800. We may assume that the tests of the chips are independent trials. Thus,

    $$X \sim \text{Binomial}(n, p),$$

    where $n = 800$ and $p = 0.001$. We want to estimate $\Pr(X \leqslant 3)$.

    Since n is large and p is very small, we may use the Poisson approximation to the binomial distribution to get

    $$\Pr(X \leqslant 3) \approx \Pr(Y \leqslant 3), \qquad \text{where } Y \sim \text{Poisson}(0.8).$$

    It then follows that

    $$\Pr(X \leqslant 3) \approx e^{-0.8} + (0.8)e^{-0.8} + \frac{(0.8)^2}{2}e^{-0.8} + \frac{(0.8)^3}{6}e^{-0.8} \doteq 0.9909;$$

    thus, the probability the lot will be accepted is about 99.09%. □

15. Last month your company sold 10,000 new watches. Past experience indicates that the probability that a new watch will need repair during its warranty period is 0.002.

    Estimate the probability that no more than 5 watches will need warranty work. Explain the reasoning leading to your estimate.

    ***Solution**:* Let X denote the number of watches out of the 10,000 that will need repair. We may assume that w given watch will needs repair is independent of the event that any watch in the batch will need repair. Thus,

    $$X \sim \text{Binomial}(n, p),$$

    where $n = 10,000$ and $p = 0.002$. We want to estimate $\Pr(X \leqslant 5)$.

    Since n is very large and p is very small, we may use the Poisson approximation to the binomial distribution to get

    $$\Pr(X \leqslant 5) \approx \Pr(Y \leqslant 5), \qquad \text{where } Y \sim \text{Poisson}(20).$$

It then follows that

$$\Pr(X \leqslant 3) \approx \sum_{k=0}^{5} \frac{(20)^k}{k!} e^{-20};$$

thus, the probability that no more than 5 watches will need warranty work is about 7.19×10^{-5}, a very small probability. $\square$