Solutions to Assignment #12

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two–dimensional subspace in $\mathbb{R}^3$; in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

***Solution**:* Let one of the unit vectors be

$$\widehat{u}_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

To find the second vector in the basis, let

$$u = v_1 + c v_2,$$

where c is determined so that

$$\langle u, v_1 \rangle = 0.$$

Thus,

$$\langle v_1 + c v_2, v_1 \rangle = 0,$$

from which we get that

$$\|v_1\|^2 + c\langle v_2, v_1 \rangle = 0,$$

or

$$6 - 3c = 0,$$

which yields that $c = 2$. It then follows that

$$u = v_1 + 2v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is orthogonal to v_1. We then let

$$\widehat{u}_2 = \frac{1}{\|u\|} u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Observe that $\widehat{u}_1$ and $\widehat{u}_2$ are linearly independent since they are orthogonal and non–zero. To see why this is the case, suppose that

$$c_1\widehat{u}_1 + c_2\widehat{u}_2 = 0.$$

Taking the inner product with $\widehat{u}_1$, we get that $c_1 = 0$; taking the inner product with $\widehat{u}_2$, we get that $c_2 = 0$. Hence the set $\{\widehat{u}_1, \widehat{u}_2\}$ is linearly independent.

Finally, since $\dim(\text{span}\{v_1, v_2\}) = 2$, it follows that $\{\widehat{u}_1, \widehat{u}_2\}$ also spans $\text{span}\{v_1, v_2\}$. Hence, $\{\widehat{u}_1, \widehat{u}_2\}$ is a basis for $\text{span}\{v_1, v_2\}$. □

2. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3x - 2y + z = 0 \right\}$. Find a non–zero vector v in $\mathbb{R}^3$ which is orthogonal to every vector in W; that is, $v \neq \mathbf{0}$ and

$$\langle v, w\rangle = 0 \quad \text{for all} \quad w \in W.$$

***Solution**:* We have seen that the set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for W (see Problem 1 in Assignment #11). We need to find a non–zero vector, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, which is orthogonal to both vectors in B; that is, the coordinates of v satisfy

$$\left\{ \begin{array}{rcl} a - 3c & = & 0 \\ b + 2c & = & 0. \end{array} \right. \tag{1}$$

Solving for the leading variables in system (1) and setting $c = t$, where t is an arbitrary parameter, we obtain the solutions

$$\left\{ \begin{array}{rcl} a & = & 3t \\ b & = & -2t \\ c & = & t. \end{array} \right. \tag{2}$$

Since, we are looking for a non–zero vector which is orthogonal to W, we can take $t = 1$ in (2) to get $v = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. □

3. Let $u_1, u_2, \ldots, u_n$ be unit vectors in $\mathbb{R}^n$ which are mutually orthogonal; that is,

$$\langle u_i, u_j \rangle = 0 \quad \text{for} \quad i \neq j.$$

Prove that the set $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, and that, for any $v \in \mathbb{R}^n$,

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle \ u_i.$$

***Solution**:* Since $\{u_1, u_2, \ldots, u_n\}$ has n vectors and $\dim(\mathbb{R}^n) = n$, it suffices to prove that the set is linearly independent. Thus, consider the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = \mathbf{0}.$$

Taking the dot product with u_j, for j in $\{1, 2, \ldots, n\}$, on both sides, and using the bilinearity property of the Euclidean inner product, we get

$$c_1 \langle u_1, u_j \rangle + c_2 \langle u_2, u_j + \cdots + c_j \langle u_j, u_j \rangle + \cdots + c_n \langle u_n, u_j \rangle = 0,$$

which implies that $c_j = 0$, since the u_i's are mutually orthogonal and $\langle u_j, u_j \rangle = 1$. Consequently,

$$c_1 = c_2 = c_3 = \cdots = c_n = 0.$$

It then follows that $\{u_1, u_2, \ldots, u_n\}$ is linearly independent.

Next, since $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, given any vector v in $\mathbb{R}^n$, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$v = c_1 u_2 + c_2 u_2 + \cdots + c_n u_n = \sum_{i=1}^{n} c_i u_i.$$

Taking the dot product with u_j on both sides we get that

$$\langle v, u_j \rangle = c_j,$$

since $\langle u_i, u_j \rangle = 0$ when $j \neq i$ and $\langle u_j, u_j \rangle = \|u_j\|^2 = 1$ for all i and j in $\{1, 2, \ldots, n\}$. Hence,

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle \ u_i.$$

$\square$

4. The Euclidean inner product of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors v, v_1, v_2 and w in $\mathbb{R}^n$,

   (i) $\langle v, w\rangle = \langle w, v\rangle$,

   (ii) $\langle c_1v_1 + c_2v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle$, and

   (iii) $\langle v, v\rangle \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\langle v, v\rangle = 0$ if and only if v is the zero vector.

   Use these properties of the the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$:

   (a) $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \mathbf{0}$.

   Solution: By the definition, $\|v\| = \sqrt{\langle v, v\rangle}$, of the norm and the positive definiteness of the Euclidean inner product, we see that $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$. Furthermore, $\|v\| = 0$ if and only if $v = \mathbf{0}$. □

   (b) For a scalar c, $\|cv\| = |c|\|v\|$.

   Solution: Use the definition of the Euclidean norm and the bilinearity of the inner product to write $\|cv\|^2 = \langle cv, cv\rangle = c^2\langle v, v\rangle$. Thus,

   $$\|cv\|^2 = c^2\|v\|^2.$$

   Taken square roots on both sides we get

   $$\|cv\| = \sqrt{c^2}\|v\| = |c|\|v\|.$$

   □

5. The Cauchy-Schwarz inequality for any vectors v and w in $\mathbb{R}^n$ states that

   $$|\langle v, w\rangle| \leqslant \|v\|\|w\|.$$

   Use this inequality to derive the triangle inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$\|v + w\| \leqslant \|v\| + \|w\|.$$

   (*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

***Solution**:* Expand $\|v+w\|^2$ using the properties of the inner product to get

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w\rangle \\ &= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\ &= \|v\|^2 + 2\langle v,w\rangle + \|w\|^2 \\ &\leqslant \|v\|^2 + 2|\langle v,w\rangle| + \|w\|^2. \end{aligned}$$

It then follows by the Cauchy–Schwarz inequality that

$$\|v+w\|^2 \leqslant \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Taking square roots yields the triangle inequality

$$\|v+w\| \leqslant \|v\| + \|w\|.$$

□