Assignment #14

Due on Friday, October 31, 2014

Read Section 3.1, on *Vector Space Structure in* $\mathbb{M}(m,n)$, in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.2, on *Matrix Algebra,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions

(Identity matrix). The $n \times n$ matrix $I = [\delta_{ij}]$ defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for $1 \leqslant i, j \leqslant n$ is called the **identity** matrix in $\mathbb{M}(n,n)$.

Do the following problems

1. Let $\mathbb{C}(2,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}(2,2) \;\middle|\; d = a \text{ and } c = -b \right\}$. It was shown in Problem 1 in Assignment #13 that $\mathbb{C}(2,2)$ is a subspace of $\mathbb{M}(2,2)$.

   (a) Prove that $\mathbb{C}(2,2) = \text{span}\{I, J\}$, where

   $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   (b) Observe that $J^2 = JJ = -I$ and compute J^n, where $n = 1, 2, 3, \ldots$

2. Let $\mathbb{C}(2,2)$ be as in Problem 1.

   (a) Prove that if Z_1 and Z_2 are two matrices in $\mathbb{C}(2,2)$, then $Z_1Z_2 \in \mathbb{C}(2,2)$; that is, $\mathbb{C}(2,2)$ is closed under matrix multiplication.

   (b) Let Z_1 and Z_2 be two matrices in $\mathbb{C}(2,2)$. Prove that $Z_1Z_2 = Z_2Z_1$; that is, matrix multiplication in $\mathbb{C}(2,2)$ is commutative.

   (c) Give the coordinates of Z_1, Z_2 and Z_1Z_2 relative to the basis $\mathcal{B} = \{I, J\}$ of $\mathbb{C}(2,2)$.

3. Let $\mathbb{C}(2,2)$ be as in Problem 1.

   (a) Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a^2 + b^2 \neq 0$. Prove that there exists a matrix Z in $\mathbb{C}(2,2)$ such that
   $$AZ = I.$$
   Suggestion: Write $Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y denote real numbers, compute AZ and find x and y so that $AZ = I$. Consider separately the cases $a \neq 0$ and $a = 0$. Observe that, since $a^2 + b^2 \neq 0$, if $a = 0$, then $b \neq 0$.

   (b) Put $\mathcal{B} = \{I, J\}$ and find the coordinates of A and Z relative to $\mathcal{B}$.

4. Consider the system of linear equations
   $$\left\{ \begin{array}{rcr} 2x_1 - x_2 - 3x_3 & = & 4 \\ x_1 + x_2 + x_3 & = & -2 \\ x_1 + 2x_2 + 3x_3 & = & 5. \end{array} \right. \qquad (1)$$

   (a) Find a 3×3 matrix A and 3×1 matrices x and b (that is, x and y are vectors in $\mathbb{R}^3$) so that the system in (1) can be expressed as the matrix equation
   $$Ax = b.$$

   (b) Let C denote the matrix $\begin{pmatrix} 1 & -3 & 2 \\ -2 & 9 & -5 \\ 1 & -5 & 3 \end{pmatrix}$, and compute the products CA, AC and Cb.

   (c) Prove that $x = Cb$ is the unique solution to the system in (1).

5. Find matrices A and B in $\mathbb{M}(2,2)$ that have no entries equal to 0, but such that
   $$AB = O,$$
   where O denotes the 2×2 zero matrix.

   Explain why, in this case, it is impossible to find 2×2 matrix C such that $CA = I$, where I denotes the 2×2 identity matrix.