

Assignment #18

Due on Friday, November 14, 2014

Read Section 4.3 on *Matrix Representation of Linear Functions* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 4.4, on *Compositions*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 2.2 on *Linear Transformations between Finite Dimensional Vector Spaces* in Damiano and Little (pp. 73–81)

Read Section 2.5 on *Composition of Linear Transformations* in Damiano and Little (pp. 106–111)

Do the following problems

1. Given two vector-valued functions, T and R , from \mathbb{R}^n to \mathbb{R}^m , we can define the sum, $T + R$, of T and R by

$$(T + R)(v) = T(v) + R(v) \quad \text{for all } v \in \mathbb{R}^n.$$

- (a) Verify that, if both T and R are linear, then so is $T + R$.
 - (b) Explain how to define the scalar multiple $aT: \mathbb{R}^n \rightarrow \mathbb{R}^m$ of a vector valued function, $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, where a is a scalar and verify that if T is linear then so is aT .
2. The **identity** function, $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$, is defined by $I(v) = v$ for all $v \in \mathbb{R}^n$.
 - (a) Verify that $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation.
 - (b) Give the matrix representation of I relative to the standard basis in \mathbb{R}^n .
 - (c) Compute the null space, \mathcal{N}_I , and image, \mathcal{I}_I , of I .
 3. The **zero** function, $O: \mathbb{R}^n \rightarrow \mathbb{R}^m$, is defined by $O(v) = \mathbf{0}$ for all $v \in \mathbb{R}^n$.
 - (a) Verify that $O: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation.
 - (b) Give the matrix representation of O relative to the standard bases in \mathbb{R}^n and \mathbb{R}^m .
 - (c) Compute the null space, \mathcal{N}_O , and image, \mathcal{I}_O , of O .

4. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote a linear function and let $M_T \in \mathbb{M}(m, n)$ be its matrix representation with respect to the standard bases in \mathbb{R}^n and \mathbb{R}^m .

- (a) Prove that the null space of T , \mathcal{N}_T , is the null space of the matrix M_T .
- (b) Prove that the image of T , \mathcal{I}_T , is the span of the columns of the matrix M_T .

5. If $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function, we can define the **iterates**, T^k , of T , where k is a positive integer, as follows:

$$T^2 = T \circ T;$$

That is, T is the composition of T with itself. Next, define

$$T^3 = T^2 \circ T$$

and so on. More precisely, once we have defined T^{k-1} for $k > 1$, we can define T^k by

$$T^k = T^{k-1} \circ T.$$

- (a) Prove that if T is a linear function from \mathbb{R}^n to \mathbb{R}^n , then so are the functions T^k for $k = 1, 2, \dots$
- (b) Prove that T^m and T^k commute with each other; that is,

$$T^m \circ T^k = T^k \circ T^m,$$

where k and m are positive integers.

- (c) Given $v \in \mathbb{R}^n$, prove that the set

$$\{v, T(v), T^2(v), \dots, T^n(v)\}$$

is linearly dependent.