Solutions to Assignment #1

1. Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (a) Write the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of v_1 and v_2. That is, find scalars c_1 and c_2 such that $v = c_1 v_1 + c_2 v_2$.

   Solution: Find scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which may be written as

$$\begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We then get the linear equations in c_1 and c_2:

$$c_1 + 2c_2 = 1$$

and

$$2c_1 + c_2 = 1.$$

Solving for c_1 in the first equation and substituting into the second equation we obtain the equation

$$2(1 - 2c_2) + c_2 = 1,$$

which can be solved for c_2 to obtain that

$$c_2 = \frac{1}{3}.$$

We then have that

$$c_1 = \frac{1}{3}.$$

Hence,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} v_1 + \frac{1}{3} v_2.$$

□

(b) Write any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ as a linear combination of v_1 and v_2.

***Solution**:* Proceed as in the previous part to find scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

which may be written as

$$\begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We then get the linear equations in c_1 and c_2:

$$c_1 + 2c_2 = x$$

and

$$2c_1 + c_2 = y.$$

Solving for c_1 in the first equation and substituting into the second equation we obtain the equation

$$2(x - 2c_2) + c_2 = y,$$

which can be solved for c_2 to obtain that

$$c_2 = \frac{2}{3}x - \frac{1}{3}y.$$

We then have that

$$c_1 = -\frac{1}{3}x + \frac{2}{3}y.$$

Consequently,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(-\frac{1}{3}x + \frac{2}{3}y\right) v_1 + \left(\frac{2}{3}x - \frac{1}{3}y\right) v_2.$$

□

2. In this problem, a, b, c and d denote scalars, and elements in $\mathbb{R}^n$ are expressed as row vectors for convenience.

(a) Find a, b and c so that $a(2,3,-1)+b(1,0,4)+c(-3,1,2)=(7,2,5)$, if possible.

Solution: Write

$$(2a,3a,-a)+(b,0,4b)+(-3c,c,2c)=(7,2,5),$$

or

$$(2a+b-3c,3a+c,-a+4b+2c)=(7,2,5),$$

which leads to the system of equations

$$\begin{cases} 2a+b-3c &= 7 \\ 3a+c &= 2 \\ -a+4b+2c &= 5 \end{cases}$$

We can solve for c in the second equation and substitute in the first equation to obtain the two equations in a and b:

$$\begin{cases} 11a+b &= 13 \\ -7a+4b &= 1. \end{cases}$$

We can then solve for b in the first equation and substitute in the second to get that

$$a=1.$$

Therefore,

$$b=2,$$

and

$$c=-1.$$

□

(b) Find a, b, c and d so that

$$a(1,0,0,0,0)+b(1,1,0,0,0)+c(1,1,1,0,0)+d(1,1,1,1,0)=(8,5,-2,3,0),$$

if possible.

Solution: Write

$$(a,0,0,0,0)+(b,b,0,0,0)+(c,c,c,0,0)+(d,d,d,d,0)=(8,5,-2,3,0),$$

or

$$(a+b+c+d,b+c+d,c+d,d,0)=(8,5,-2,3,0),$$

which yields the system of linear equations

$$\begin{cases} a+b+c+d &= 8 \\ b+c+d &= 5 \\ c+d &= -2 \\ d &= 3. \end{cases}$$

We then get that $d = 3$, $c = -5$, $b = 7$ and $a = 3$. □

3. Show that it is impossible to find scalars a, b, c and d so that

$$a(1,0,0,0,0) + b(1,1,0,0,0) + c(1,1,1,0,0) + d(1,1,1,1,0) = (8,5,-2,3,1).$$

Solution: Proceeding as in part (b) of the previous problem, we obtain the system of linear equations

$$\begin{cases} a+b+c+d &= 8 \\ b+c+d &= 5 \\ c+d &= -2 \\ d &= 3 \\ 0 &= 1, \end{cases}$$

where the last equation is impossible. We therefore conclude that that it is impossible to find scalars a, b, c and d so that

$$a(1,0,0,0,0)+b(1,1,0,0,0)+c(1,1,1,0,0)+d(1,1,1,1,0) = (8,5,-2,3,1).$$

□

4. The equation $5x - 2y + 8z = 0$ describes a plane in $\mathbb{R}^3$. Let (a_1, a_2, a_3) be any point on the plane; that is $5a_1 - 2a_2 + 8a_3 = 0$. Show that the vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is a linear combination of the vectors

$$\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}.$$

Solution: Since (a_1, a_2, a_3) be any point on the plane, it follows that

$$5a_1 - 2a_2 + 8a_3 = 0,$$

which can be solved for a_2 as follows

$$a_2 = \frac{5}{2}a_1 + 4a_3,$$

where a_1 and a_3 are arbitrary. We can therefore set $a_1 = 2t$ and $a_3 = s$, where t and s are arbitrary scalars.
We then get that

$$\begin{cases} a_1 &= 2t \\ a_2 &= 5t + 4s \\ a_3 &= s. \end{cases}$$

We therefore have that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2t \\ 5t + 4s \\ s \end{pmatrix} = \begin{pmatrix} 2t \\ 5t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4s \\ s \end{pmatrix} = t\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},$$

which was to be shown. □

5. Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2, x_3) = c_1(2, 5, 0) + c_2(0, 4, 1)\}$ show that if $(x, y, z) \in W$, then $5x - 2y + 8z = 0$. What can you conclude from this and the statement in problem 4?

Solution: Let $(x, y, z) \in W$; then,

$$(x, y, z) = c_1(2, 5, 0) + c_2(0, 4, 1)$$

for scalars c_1 and c_2.
We then have that

$$(2c_1, 5c_1, 0) + (0, 4c_2, c_2) = (x, y, z)$$

or

$$(2c_1, 5c_1 + 4c_2, c_2) = (x, y, z),$$

which leads to the system of equations

$$\begin{cases} 2c_1 &= x \\ 5c_1 + 4c_2 &= y \\ c_2 &= z. \end{cases}$$

Solving for c_1 and c_2 in the first and third equations, respectively, and substituting them in the second yields

$$5\left(\frac{x}{2}\right) + 4z = y.$$

Multiplying this equation by 2 and rearranging, yields $5x - 2y + 8z = 0$, which was to be shown.

Combining this result with the result in the previous problem yields that W, the span of the vectors $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$, is the plane given by the equation $5x - 2y + 8z = 0$. $\square$