Assignment #21

Due on Monday, November 24, 2014

Read Section 4.3 on *Matrix Representation of Linear Functions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.4, on *Compositions,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 2.7 on *Change of Basis* in Damiano and Little (pp. 122–127)

Background and Definitions

- **Similar Matrices**. Let A and B denote $n \times n$ matrices. We say that A and B are similar if and only if there exits and invertible $n \times n$ matrix Q such that

$$B = Q^{-1} A Q.$$

Do the following problems

1. Let $R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation around the origin in the counterclockwise through an angle θ. Let $\mathcal{B} = \{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

   Give the matrix representation for R_θ relative to B; that is, compute $[R_\theta]_{\mathcal{B}}^{\mathcal{B}}$.

2. Let R_θ be as in Problem 1 and let $\mathcal{E}$ denote the standard basis in $\mathbb{R}^2$. Compute the matrix representations $[R_\theta]_{\mathcal{E}}^{\mathcal{B}}$ and $[R_\theta]_{\mathcal{B}}^{\mathcal{E}}$.

3. The set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

   is a basis for $\mathbb{R}^3$. Let T denote a linear transformation satisfying

$$T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

   Compute M_T, the matrix representation of T relative to the standard basis in $\mathbb{R}^3$.

4. Let A and B denote $n \times n$ matrices. Assume that A and B are similar. Prove that there exists a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ and bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{R}^n$ such that
$$A = [T]_{\mathcal{B}}^{\mathcal{B}} \quad \text{and} \quad B = [T]_{\mathcal{B}'}^{\mathcal{B}'}.$$

5. The set $\mathcal{B} = \{v_1, v_2\}$, where
$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
is a basis for $\mathbb{R}^2$. Let $I\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity map. Compute the matrix representations $[I]_{\mathcal{E}}^{\mathcal{B}}$ and $[I]_{\mathcal{B}}^{\mathcal{E}}$, where $\mathcal{E}$ denotes the standard basis in $\mathbb{R}^2$.