Solutions to Assignment #21

1. Let $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation around the origin in the counterclockwise through an angle θ. Let $\mathcal{B} = \{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Give the matrix representation for R_θ relative to $\mathcal{B}$; that is, compute $[R_\theta]_{\mathcal{B}}^{\mathcal{B}}$.

***Solution**:* First, note that $R_\theta(v) = M_T v$ for all $v \in \mathbb{R}^2$, where

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The matrix representation of R_θ relative to $\mathcal{B}$ is given by

$$[R_\theta]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} [R_\theta(v_1)]_{\mathcal{B}} & [R_\theta(v_2)]_{\mathcal{B}} \end{bmatrix} \tag{1}$$

Thus, we compute $R_\theta(v_1)$ and $R_\theta(v_2)$ and their coordinates relative to $\mathcal{B}$, $[R_\theta(v_1)]_{\mathcal{B}}$ and $[R_\theta(v_2)]_{\mathcal{B}}$, respectively.

Compute

$$R_\theta(v_1) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{pmatrix}$$

Next, find c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = R_\theta(v_1),$$

or

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{pmatrix} \tag{2}$$

We can solve the equation in (2) by multiplying on both sides by

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} = \frac{1}{-5} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3}$$

Thus,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{pmatrix},$$

or

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}.$$

We therefore get that

$$[R_\theta(v_1)]_{\mathcal{B}} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}. \qquad (4)$$

Similarly, to find $[R_\theta(v_2)]_{\mathcal{B}}$, first compute

$$R_\theta(v_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos\theta + 2\sin\theta \\ \sin\theta - 2\cos\theta \end{pmatrix},$$

and then compute

$$[R_\theta(v_2)]_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \cos\theta + 2\sin\theta \\ \sin\theta - 2\cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \qquad (5)$$

Finally, combining (1), (4) and (5) yields

$$[R_\theta]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

□

2. Let R_θ be as in Problem 1 and let $\mathcal{E}$ denote the standard basis in $\mathbb{R}^2$. Compute the matrix representations $[R_\theta]_{\mathcal{E}}^{\mathcal{B}}$ and $[R_\theta]_{\mathcal{B}}^{\mathcal{E}}$.

***Solution**:* First, we compute

$$[R_\theta]_{\mathcal{E}}^{\mathcal{B}} = \left[[R_\theta(e_1)]_{\mathcal{B}} \quad [R_\theta(e_2)]_{\mathcal{B}}\right], \qquad (6)$$

where

$$R_\theta(e_1) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

and

$$R_\theta(e_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

As in Problem 1, we compute the coordinates of $R_\theta(e_1)$ and $R_\theta(e_2)$ by multiplying by the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ in (3). We get

$$[R_\theta(e_1)]_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} (2\cos\theta + \sin\theta)/5 \\ (\cos\theta - 2\sin\theta)/5 \end{pmatrix}, \qquad (7)$$

and

$$[R_\theta(e_2)]_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\theta - 2\sin\theta)/5 \\ (-\sin\theta - 2\cos\theta)/5 \end{pmatrix}. \qquad (8)$$

Combining (6), (7) and (8) then yields

$$[R_\theta]_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} (2\cos\theta + \sin\theta)/5 & (\cos\theta - 2\sin\theta)/5 \\ (\cos\theta - 2\sin\theta)/5 & (-\sin\theta - 2\cos\theta)/5 \end{pmatrix}.$$

Next, we compute

$$[R_\theta]_{\mathcal{B}}^{\mathcal{E}} = \left[[R_\theta(v_1)]_{\mathcal{E}} \quad [R_\theta(v_2)]_{\mathcal{E}} \right],$$

where

$$R_\theta(v_1) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{pmatrix},$$

and

$$R_\theta(v_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos\theta + 2\sin\theta \\ \sin\theta - 2\cos\theta \end{pmatrix}.$$

Consequently,

$$[R_\theta]_{\mathcal{B}}^{\mathcal{E}} = \begin{pmatrix} 2\cos\theta - \sin\theta & \cos\theta + 2\sin\theta \\ 2\sin\theta + \cos\theta & \sin\theta - 2\cos\theta \end{pmatrix}.$$

□

3. The set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Let T denote a linear transformation satisfying

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Compute M_T, the matrix representation of T relative to the standard basis in $\mathbb{R}^3$.

***Solution**:* Call the vectors in $\mathcal{B}$ v_1, v_2 and v_3, respectively, so that

$$T(v_1) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad T(v_2) = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad T(v_3) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

Next, compute the coordinates of e_1, e_2 and e_3 relative to $\mathcal{B}$.

For e_1 we solve

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = e_1,$$

or

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

The system in (10) can be solved by multiplying on both sides on the left by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1},$$

where the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ can be obtained by performing Gaussian elimination to the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

This leads to

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{11}$$

Thus, the solution of the system in (10) is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consequently,

$$e_1 = v_3. \tag{12}$$

Similar calculations show that

$$e_2 = v_2 - v_3. \tag{13}$$

and

$$e_3 = v_1 - v_2. \tag{14}$$

Applying the linear transformation T to the expressions in (12), (13) and (14), and using (9) then yields

$$T(e_1) = T(v_3) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

$$T(e_2) = T(v_2) - T(v_3) = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix},$$

and

$$T(e_3) = T(v_1) - T(v_2) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

We then have that

$$M_T = \begin{pmatrix} -1 & 4 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

□

4. Let A and B denote $n \times n$ matrices. Assume that A and B are similar. Prove that there exists a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ and bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{R}^n$ such that

$$A = [T]_{\mathcal{B}}^{\mathcal{B}} \quad \text{and} \quad B = [T]_{\mathcal{B}'}^{\mathcal{B}'}.$$

***Solution**:* Assume that B is similar to A; then, there exists an invertible $n \times n$ matrix Q such that

$$B = Q^{-1}AQ.$$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by

$$T(v) = Av, \quad \text{for all } v \in \mathbb{R}^n.$$

Then,

$$A = M_T = [T]_{\mathcal{E}}^{\mathcal{E}},$$

where $\mathcal{E}$ denotes the standard basis in $\mathbb{R}^n$.

Next, write Q in terms of its columns

$$Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix},$$

so that the set $\mathcal{B}' = \{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, since Q is invertible. We then have that

$$Q = [I]_{\mathcal{B}'}^{\mathcal{E}},$$

where $I\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, is the change of bases matrix from $\mathcal{B}'$ to $\mathcal{E}$. We also have that

$$Q^{-1} = [I]_{\mathcal{E}}^{\mathcal{B}'}$$

is the change of bases matrix from $\mathcal{E}$ to $\mathcal{B}$. Consequently,

$$B = Q^{-1}AQ = [I]_{\mathcal{E}}^{\mathcal{B}'}[T]_{\mathcal{E}}^{\mathcal{E}}[I]_{\mathcal{B}'}^{\mathcal{E}} = [T]_{\mathcal{B}'}^{\mathcal{B}'},$$

which was to be shown for the case $\mathcal{B} = \mathcal{E}$. □

5. The set $\mathcal{B} = \{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

is a basis for $\mathbb{R}^2$. Let $I\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity map. Compute the matrix representations $[I]_{\mathcal{E}}^{\mathcal{B}}$ and $[I]_{\mathcal{B}}^{\mathcal{E}}$, where $\mathcal{E}$ denotes the standard basis in $\mathbb{R}^2$.

***Solution**:* We first compute

$$\begin{aligned} [I]_{\mathcal{B}}^{\mathcal{E}} &= \begin{bmatrix} [I(v_1)]_{\mathcal{E}} & I(v_2)]_{\mathcal{E}} \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

To compute $[I]_{\mathcal{E}}^{\mathcal{B}}$, compute the inverse of $[I]_{\mathcal{B}}^{\mathcal{A}}$; thus,

$$\begin{aligned} [I]_{\mathcal{E}}^{\mathcal{B}} &= ([I]_{\mathcal{B}}^{\mathcal{E}})^{-1} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}. \end{aligned}$$

□