Solutions to Assignment #2

1. Consider the vectors v_1, v_2 and v_3 in $\mathbb{R}^3$ given by

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}.$$

Show that $v_3 \in \text{span}\{v_1, v_2\}$.

***Solution**:* We need to find scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = v_3,$$

or

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$$

or

$$\begin{pmatrix} c_1 + 2c_2 \\ 2c_1 - 3c_2 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}.$$

This leads to the system of equations

$$\begin{cases} c_1 + 2c_2 & = & 0 \\ 2c_1 - 3c_2 & = & 7 \\ -c_1 + c_2 & = & -3. \end{cases}$$

Solving for c_1 in the first equation and substituting into the second equation yields

$$-7c_2 = 7,$$

from which we get that

$$c_2 = -1.$$

We then get that $c_1 = 2$ from the first equation. Note that $c_1 = 2$ and $c_2 = -1$ are consistent with the third equation. It then follows that

$$v_3 = 2v_1 - v_2,$$

and therefore $v_3 \in \text{span}\{v_1, v_2\}$. □

2. Let v_1, v_2 and v_3 be as in Problem 1 above. Use the result of Problem 1 to show that
$$\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\}.$$
Note: You need to show that one span is a subset of the other, and conversely, the other is a subset of the one.

***Solution**:* To see that $\text{span}\{v_1, v_2, v_3\} \subseteq \text{span}\{v_1, v_2\}$, let v be in $\text{span}\{v_1, v_2, v_3\}$. Then,
$$v = c_1v_1 + c_2v_2 + c_3v_3,$$
for some scalars c_1, c_2 and c_3. By the result of Problem 1, $v_3 = 2v_1 - v_2$, so that
$$\begin{aligned} v &= c_1v_1 + c_2v_2 + c_3(2v_1 - v_2) \\ &= c_1v_1 + c_2v_2 + 2c_3v_1 - c_3v_2 \\ &= (c_1 + 2c_3)v_1 + (c_2 - c_3)v_2, \end{aligned}$$
which displays v as a linear combination of v_1 and v_2; that is, v is in $\text{span}\{v_1, v_2\}$. Thus, we have shown that
$$v \in \text{span}\{v_1, v_2, v_3\} \Rightarrow v \in \text{span}\{v_1, v_2\};$$
that is, $\text{span}\{v_1, v_2, v_3\} \subseteq \text{span}\{v_1, v_2\}$.
Next, we show the reverse inclusion: $\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3\}$. Let $v \in \text{span}\{v_1, v_2\}$. Then,
$$v = c_1v_1 + c_2v_2$$
for some scalars c_1 and c_2, so that
$$v = c_1v_1 + c_2v_2 + 0 \cdot v_3;$$
that is, v is also a linear combination of v_1, v_2 and v_3. Consequently,
$$v \in \text{span}\{v_1, v_2\} \Rightarrow v \in \text{span}\{v_1, v_2, v_3\},$$
or $\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3\}$.
We therefore conclude that $\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\}$. □

3. Let v_1 and v_2 be as in Problem 1 above. Show that $\text{span}\{v_1, v_2\}$ is a plane through the origin in $\mathbb{R}^3$ and give the equation of the plane.

***Solution**:* Consider an arbitrary element, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, in $\text{span}\{v_1, v_2\}$. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 v_1 + c_2 v_2$$

for scalars c_1 and c_2. That is,

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or

$$\begin{pmatrix} c_1 + 2c_2 \\ 2c_1 - 3c_2 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We then get the system of equations

$$\begin{cases} c_1 + 2c_2 & = & x \\ 2c_1 - 3c_2 & = & y \\ -c_1 + c_2 & = & z. \end{cases}$$

Solving for c_1 in the first equation and substituting into the second and third equations leads to the system of two equations

$$\begin{cases} 7c_2 & = & 2x - y \\ 3c_2 & = & x + z. \end{cases}$$

We then get that

$$\frac{2x - y}{7} = \frac{x + z}{3},$$

from which we get the equation

$$x + 3y + 7z = 0,$$

which is the equation of a plane in $\mathbb{R}^3$ containing the vectors v_1 and v_2. Denoting the plane by Q, we see that we have just shown that

$$\text{span}\{v_1, v_2\} \subseteq Q.$$

To show that Q is a subset of span$\{v_1, v_2\}$, we need to show that any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ whose coordinates satisfy

$$x + 3y + 7z = 0$$

must be a linear combination of v_1 and v_2. To see why this is so, solve for x in terms of y and z to get

$$x = -3y - 7z.$$

Setting $y = t$ and $z = s$ to be arbitrary parameters, we see then that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix},$$

which shows that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in the span of the vectors $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$. Hence, it suffices to show that both $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$ are in span$\{v_1, v_2\}$. Observe that

$$\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = -v_1 - v_2$$

and

$$\begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix} = -3v_1 - 2v_2.$$

Thus, both $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$ are in span$\{v_1, v_2\}$. We therefore conclude that, if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in Q$, then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$ span$\{v_1, v_2\}$. Consequently, the span of v_1 and v_2 is the plane in $\mathbb{R}^3$ determined by the equation $x + 3y + 7z = 0$. $\square$

4. Let v_1 and v_2 be as in Problem 1 above. Find a vector in $\mathbb{R}^3$ which is not in the span of v_1 and v_2. Call the vector v_4 and show that

$$\text{span}\{v_1, v_2, v_4\} = \mathbb{R}^3.$$

***Solution**:* We saw in the solution to the previous problem that $\text{span}\{v_2, v_2\}$ is the plane in $\mathbb{R}^3$ given by the equation $x + 3y + 7z = 0$. Thus, any vector in $\mathbb{R}^3$ whose components do not satisfy the equation is not in the span of v_1 and v_2. In particular the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in $\text{span}\{v_1, v_2\}$. Let

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We show that

$$\text{span}\{v_1, v_2, v_4\} = \mathbb{R}^3.$$

Observe first that

$$\text{span}\{v_1, v_2, v_4\} \subseteq \mathbb{R}^3$$

since v_1, v_2 and v_4 are vectors in $\mathbb{R}^3$. Hence, it suffices to show that

$$\mathbb{R}^3 \subseteq \text{span}\{v_1, v_2, v_4\}.$$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an arbitrary vector in $\mathbb{R}^3$. We would like to find scalars c_1, c_2 and c_3 such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This leads to the system of equations

$$\begin{cases} c_1 + 2c_2 & = & x \\ 2c_1 - 3c_2 & = & y \\ -c_1 + c_2 + c_3 & = & z. \end{cases}$$

Solving for c_1 and c_2 in the first two equations yields

$$\begin{aligned} c_1 &= \frac{3}{7}x + \frac{2}{7}y \\ c_2 &= \frac{2}{7}x - \frac{1}{7}y. \end{aligned}$$

Substituting these into the third equation and solving for c_3 then yields

$$c_3 = \frac{1}{7}x + \frac{3}{7}y + z.$$

Hence, every vector in $\mathbb{R}^3$ can be written as a linear combination of v_1, v_2 and v_4; in fact,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{3}{7}x + \frac{2}{7}y\right) v_1 + \left(\frac{2}{7}x - \frac{1}{7}y\right) v_2 + \left(\frac{1}{7}x + \frac{3}{7}y + z\right) v_4.$$

We therefore conclude that the set $\{v_1, v_2, v_4\}$ spans $\mathbb{R}^3$. □

5. Let v_1 and v_2 be as in Problem 1 above. Determine, if possible, a value of c for which the vector

$$\begin{pmatrix} 4 \\ 1 \\ c \end{pmatrix}$$

lies in span$\{v_1, v_2\}$. How many values of c with that property are there?

***Solution**:* Look for scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} 4 \\ 1 \\ c \end{pmatrix}.$$

This leads to the system of equations

$$\begin{cases} c_1 + 2c_2 &= 4 \\ 2c_1 - 3c_2 &= 1 \\ -c_1 + c_2 &= c. \end{cases}$$

Solving for c_1 and c_2 in the first two equations yields

$$\begin{aligned} c_1 &= 2 \\ c_2 &= 1. \end{aligned}$$

It then follows from the third equation that $c = -1$. Thus, c must be -1 in order for the vector $\begin{pmatrix} 4 \\ 1 \\ c \end{pmatrix}$ to be in the span of v_1 and v_2. There is only one value of c for which this is the case. $\square$

Alternate Solution: We can also solve this problem by using the characterization of $\text{span}\{v_1, v_2\}$ as the plane determined by the equation $x + 3y + 7z = 0$. If the point $\begin{pmatrix} 4 \\ 1 \\ c \end{pmatrix}$ is in the plane, then $x = 4$, $y = 1$, and $z = c$. It then follows that

$$4 + 3 + 7c = 0,$$

from which we get that $c = -1$. $\square$