Solutions to Assignment #3

1. Consider the vectors v_1, v_2 and v_3 in $\mathbb{R}^3$ given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}.$$

(a) If possible, write the vector v_3 as a linear combination of v_1 and v_2.

Solution: Consider the equation

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}.$$

This leads to the system

$$\begin{cases} c_1 + 2c_2 & = & 0 \\ 5c_2 & = & -4 \\ -c_1 + c_2 & = & 3. \end{cases}$$

Solving for c_1 and c_2 in the first two equations leads to

$$\begin{aligned} c_2 &= -4/5 \\ c_1 &= 8/5. \end{aligned}$$

Substituting for these into the third equation leads to

$$-12/5 = 3,$$

which is impossible. Thus, there are no scalars c_1 and c_2 such that $v_3 = c_1 v_1 + c_2 v_2$; in other words, it is impossible to write the vector v_3 as a linear combination of v_1 and v_2. □

(b) Determine whether the set $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$.

Solution: We need to show that any vector, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, in $\mathbb{R}^3$ can be written as a linear of the vectors v_1, v_2 and v_3. Thus, we look for scalars c_1, c_2 and c_3 such that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1}$$

This leads to the system

$$\begin{cases} c_1 + 2c_2 & = & x \\ 5c_2 - 4c_3 & = & y \\ -c_1 + c_2 + 3c_3 & = & z. \end{cases} \tag{2}$$

Solving for c_1 in the first equation in (2) and substituting for c_1 in the third equation leads to the two equations

$$\begin{cases} 5c_2 - 4c_3 & = & y \\ 3c_2 + 3c_3 & = & x + z. \end{cases}$$

Solving this system yields

$$\begin{aligned} c_2 &= \frac{4}{27}x + \frac{4}{9}y + \frac{5}{27}z \\ c_3 &= \frac{5}{27}x - \frac{1}{9}y + \frac{5}{27}z. \end{aligned}$$

It then follows from the first equation in (2) that

$$c_1 = \frac{19}{27}x - \frac{8}{9}y - \frac{10}{27}z.$$

Consequently, there exist c_1, c_2 and c_3, depending on x, y and z, for which (1) holds for any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$. We therefore conclude that the set $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$. □

2. Let v_1, v_2 and v_3 be as given in the previous problem. Find a linearly independent subset of $\{v_1, v_2, v_3\}$ which spans span$\{v_1, v_2, v_3\}$.

***Solution**:* The set $\{v_1, v_2, v_3\}$ is linearly independent. To see why this is so, consider the equation

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

This leads to the system

$$\begin{cases} c_1 + 2c_2 & = & 0 \\ 5c_2 - 4c_3 & = & 0 \\ -c_1 + c_2 + 3c_3 & = & 0. \end{cases} \tag{4}$$

Solving for c_1 in the first equation and substituting for c_1 in the third equation leads to the two equations

$$\left\{\begin{array}{rcl} 5c_2 - 4c_3 & = & 0 \\ 3c_2 + 3c_3 & = & 0. \end{array}\right.$$

Solving this system yields

$$\begin{array}{rcl} c_2 & = & 0 \\ c_3 & = & 0. \end{array}$$

It then follows from the third equation in (4) that $c_1 = 0$. Consequently, equation (3) has only the trivial solution $c_1 = c_2 = c_3 = 0$. We therefore conclude that the set $\{v_1, v_2, v_3\}$ is linearly independent. Hence, $\{v_1, v_2, v_3\}$ is al linearly independent subset of itself which spans $\operatorname{span}\{v_1, v_2, v_3\}$ $\square$

3. Show that the set $\left\{\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right\}$ is a linearly independent subset of $\mathbb{R}^3$.

***Solution**:* Consider the equation

$$c_1 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

This leads to the system

$$\left\{\begin{array}{rcl} 2c_1 + 3c_2 + c_3 & = & 0 \\ 4c_1 + 2c_2 - 2c_3 & = & 0 \\ 2c_1 + 2c_3 & = & 0. \end{array}\right. \tag{6}$$

Solving for c_3 in the third equation in (6) and substituting for c_3 into the first and second equations leads to the two equations

$$\left\{\begin{array}{rcl} c_1 + 3c_2 & = & 0 \\ 6c_1 + 2c_2 & = & 0. \end{array}\right.$$

Solving this system yields

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0. \end{aligned}$$

It then follows from the third equation in (6) that $c_3 = 0$. Consequently, equation (5) has only the trivial solution $c_1 = c_2 = c_3 = 0$. We therefore conclude that the set $\left\{ \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\}$ is linearly independent. □

4. Determine whether the set $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^4$.

***Solution**:* Consider the equation

$$c_1 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (7)$$

This leads to the system

$$\left\{ \begin{aligned} 2c_1 + 2c_3 &= 0 \\ -c_1 + 2c_2 &= 0 \\ -c_2 - c_3 &= 0 \\ c_1 - 2c_2 &= 0. \end{aligned} \right. \qquad (8)$$

This system reduces to the system of two equations

$$\left\{ \begin{aligned} c_1 + c_3 &= 0 \\ -c_1 + 2c_2 &= 0 \\ -c_2 - c_3 &= 0, \end{aligned} \right. \qquad (9)$$

since the second and the fourth equations in (8) are the same equation. Solving for c_3 in the third equation in (9) and substituting into the first equation in the same system leads to

$$\left\{ \begin{aligned} c_1 - c_2 &= 0 \\ -c_1 + 2c_2 &= 0, \end{aligned} \right. \qquad (10)$$

which can be solved to yield that $c_1 = c_2 = 0$. Consequently, by the first equation in (9), $c_3 = 0$. Thus, the vector equation (7) has only the trivial solution $c_1 = c_2 = c_3 = 0$. It then follows that the set

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^4$. □

5. Show that $\left\{ \begin{pmatrix} 2 \\ 2 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right\}$ is a linearly dependent subset of $\mathbb{R}^4$. Write one of the vectors in the set as a linear combination of the other three. Show that the remaining three vectors form a linearly independent subset of $\mathbb{R}^4$.

***Solution**:* Consider the equation

$$c_1 \begin{pmatrix} 2 \\ 2 \\ 6 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (11)$$

This leads to the system

$$\left\{ \begin{aligned} 2c_1 + c_3 + c_4 &= 0 \\ 2c_1 - c_2 + 2c_3 - c_4 &= 0 \\ 6c_1 + 3c_3 + 3c_4 &= 0 \\ c_2 + 3c_3 - 2c_4 &= 0. \end{aligned} \right. \qquad (12)$$

Observe that the first and third equation in (12) are really the same equation since the third is just the first equation times 3. Solve for c_4 in the first equation in (12) and substitute into the second and fourth equations to get the system of two equations

$$\left\{ \begin{aligned} 4c_1 - c_2 + 3c_3 &= 0 \\ 4c_1 + c_2 + 5c_3 &= 0. \end{aligned} \right. \qquad (13)$$

Next, solve the first equation in (13) for $4c_1$ and substitute into the second equation to obtain

$$\left\{\begin{array}{lcl} 4c_1 - c_2 + 3c_3 & = & 0 \\ 2c_2 + 2c_3 & = & 0. \end{array}\right. \tag{14}$$

Solve for c_2 in the second equation in (14) and substitute into the first to get that

$$\left\{\begin{array}{lcl} c_1 + c_3 & = & 0 \\ c_2 + c_3 & = & 0. \end{array}\right. \tag{15}$$

We can then solve for c_1 and c_2 in terms of c_3 to obtain from (15) that

$$\left\{\begin{array}{lcl} c_1 & = & -c_3 \\ c_2 & = & -c_3. \end{array}\right. \tag{16}$$

Setting $c_3 = t$, where t is an arbitrary parameter, we obtain from (16) that

$$\left\{\begin{array}{lcl} c_1 & = & -t \\ c_2 & = & -t \\ c_3 & = & t. \end{array}\right. \tag{17}$$

Since t is arbitrary, we see that the system (12) has infinitely many solutions given by

$$\left\{\begin{array}{lcl} c_1 & = & -t \\ c_2 & = & -t \\ c_3 & = & t \\ c_4 & = & t. \end{array}\right. \tag{18}$$

In particular, we then see that the vector equation (11) has a nontrivial solution and therefore the set $\left\{ \begin{pmatrix} 2 \\ 2 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right\}$ is a linearly dependent subset of $\mathbb{R}^4$. Call the vectors in the set v_1, v_2, v_3 and v_4, respectively. Taking $t = 1$ in (18) we then get from the vector equation in (11) that

$$-v_1 - v_2 + v_3 + v_4 = \mathbf{0}.$$

We can therefore solve for v_4 in terms of v_1, v_2 and v_3:

$$v_4 = v_1 + v_2 - v_3.$$

We now show that the vectors v_1, v_2 and v_3 are linearly independent. To do this, consider the equation

$$c_1\begin{pmatrix}2\\2\\6\\0\end{pmatrix}+c_2\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}+c_3\begin{pmatrix}1\\2\\3\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{19}$$

This leads to the system

$$\left\{\begin{array}{rcl} 2c_1+c_3 & = & 0\\ 2c_1-c_2+2c_3 & = & 0\\ 6c_1+3c_3 & = & 0\\ c_2+3c_3 & = & 0.\end{array}\right. \tag{20}$$

Observe that the third equation in (20) is 3 times first; thus, the system (20) reduces to

$$\left\{\begin{array}{rcl} 2c_1+c_3 & = & 0\\ 2c_1-c_2+2c_3 & = & 0\\ c_2+3c_3 & = & 0.\end{array}\right. \tag{21}$$

Solving for c_2 in the third equation in (21) and substituting for c_2 into the second equation leads to the two equations

$$\left\{\begin{array}{rcl} 2c_1+c_3 & = & 0\\ 2c_1+5c_3 & = & 0.\end{array}\right.$$

Solving this system yields

$$\begin{array}{rcl} c_1 & = & 0\\ c_3 & = & 0.\end{array}$$

It then follows from the third equation in (21) that $c_2=0$. Consequently, equation (19) has only the trivial solution $c_1=c_2=c_3=0$. We therefore conclude that the set $\left\{\begin{pmatrix}2\\2\\6\\0\end{pmatrix},\begin{pmatrix}0\\-1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\\3\end{pmatrix}\right\}$ is linearly independent. □