Solutions to Assignment #4

1. Let $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x \geqslant 0, y \geqslant 0 \right\}$. Show that S is closed under vector addition in $\mathbb{R}^2$. Explain why S is not a subspace of $\mathbb{R}^2$.

   ***Solution**:* Let $v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $w = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be vectors in S. It then follows that $x_1, y_1, x_2, y_2 \geqslant 0$. Consequently,

   $$x_1 + x_2 \geqslant 0 \quad \text{and} \quad y_1 + y_2 \geqslant 0,$$

   which shows that

   $$v + w = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in S,$$

   and therefore S is closed under vector addition in $\mathbb{R}^2$. However, S is not a subspace of $\mathbb{R}^2$ because S is not closed under scalar multiplication; to see this, note that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$, but

   $$(-1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin S.$$

   □

2. Let $a_1, a_2, b_1, b_2, c_1, c_2$ be real constants. Let W be the solution set of the homogeneous system

   $$\begin{cases} a_1x_1 + b_1x_2 + c_1x_3 &= 0 \\ a_2x_1 + b_2x_2 + c_2x_3 &= 0. \end{cases}$$

   Prove that W is a subspace of $\mathbb{R}^3$.

   ***Solution**:* Note that W is a subset of $\mathbb{R}^3$ given by

   $$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} a_1x_1 + b_1x_2 + c_1x_3 &= 0 \\ a_2x_1 + b_2x_2 + c_2x_3 &= 0 \end{cases} \right\}$$

   First, observe that $x_1 = x_2 = x_3 = 0$ solves the system. Consequently, W is not empty.

Suppose that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ are solutions of the system. Then,

$$\left\{ \begin{array}{rcl} a_1x_1 + b_1x_2 + c_1x_3 & = & 0 \\ a_2x_1 + b_2x_2 + c_2x_3 & = & 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} a_1y_1 + b_1y_2 + c_1y_3 & = & 0 \\ a_2y_1 + b_2y_2 + c_2y_3 & = & 0. \end{array} \right.$$

Adding the first equations of the systems and the second equations yields

$$\left\{ \begin{array}{rcl} a_1(x_1 + y_1) + b_1(x_2 + y_2) + c_1(x_3 + y_3) & = & 0 \\ a_2(x_1 + y_1) + b_2(x_2 + y_2) + c_2(x_3 + y_3) & = & 0, \end{array} \right.$$

where we have used the distributive property for real numbers. It then follows that $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$ is a solution of the system, and therefore W is closed under vector addition in $\mathbb{R}^3$.

Next, suppose that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a solution of the system

$$\left\{ \begin{array}{rcl} a_1x_1 + b_1x_2 + c_1x_3 & = & 0 \\ a_2x_1 + b_2x_2 + c_2x_3 & = & 0. \end{array} \right.$$

Multiplying both equations in the system by a scalar t we obtain

$$\left\{ \begin{array}{rcl} a_1(tx_1) + b_1(tx_2) + c_1(tx_3) & = & 0 \\ a_2(tx_1) + b_2(tx_2) + c_2(tx_3) & = & 0, \end{array} \right.$$

where we have applied the distributive and associative properties for real numbers. It then follows that $\begin{pmatrix} tx_1 \\ tx_2 \\ tx_3 \end{pmatrix} \in W$, and therefore W is also closed under scalar multiplication. Hence, we conclude that W is a subspace of $\mathbb{R}^3$. □

3. Let $L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 2x + 1 \right\}$. Determine whether or not L is a subspace of $\mathbb{R}^2$.

Solution: L is not a subspace of $\mathbb{R}^2$. To see this, note that the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is in L; however, the vector $(-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is not in L since $y = -1$ and $x = 0$ do not satisfy the equation $y = 2x + 1$. □

4. Let W be a subspace of $\mathbb{R}^n$. Use the definition of subspace to prove the following statements.

   (a) If $v \in W$, then W must also contain the additive inverse of v.

   Proof: Since W is subspace of $\mathbb{R}^n$, it is closed under scalar multiplication. It then follows that, if $v \in W$, then $(-1)v \in W$; that is $-v \in W$. □

   (b) W contains the zero vector.

   Proof: Since W is a subspace, it is non–empty; therefore, it contains a vector v. By the previous part, $-v \in W$. Hence, since W is closed under vector addition, $v + (-v) \in W$, which shows that $\mathbf{0} \in W$. □

5. Given two subsets A and B of $\mathbb{R}^n$, the **intersection** of A and B, denoted by $A \cap B$, is the set which contains all vectors that are both in A and B; in symbols,

$$A \cap B = \{v \in \mathbb{R}^n \mid v \in A \text{ and } v \in B\}.$$

   (a) Prove that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

   Proof: If $x \in A \cap B$ then $x \in A$ and $x \in B$, by the definition of intersection. Thus, $x \in B$. We have therefore shown that

$$x \in A \cap B \Rightarrow x \in A,$$

   which shows that $A \cap B \subseteq A$.
   A similar argument shows that $A \cap B \subseteq B$. □

   (b) Prove that if W_1 and W_2 are two subspaces of $\mathbb{R}^n$, then the intersection $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ which is contained in both W_1 and W_2.

Proof: We first show that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

Since W_1 and W_2 are subspace of $\mathbb{R}^n$, it follows from the result in part (b) of problem 4 in this assignment that $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$. Consequently, $\mathbf{0} \in W_1 \cap W_2$, which shows that $W_1 \cap W_2$ is not empty.

Next, suppose that $v, w \in W_1 \cap W_2$. Then, $v \in W_1$ and $w \in W_1$ so that

$$v + w \in W_1$$

since W_1 is closed under vector addition. Similarly, we can show that

$$v + w \in W_2.$$

It then follows that

$$v + w \in W_1 \cap W_2,$$

and therefore $W_1 \cap W_2$ is closed under vector addition.

Finally, if $v \in W_1 \cap W_2$ and $t \in \mathbb{R}$, we have that $v \in W_1$ and $v \in W_2$ and therefore

$$tv \in W_1 \quad \text{and} \quad tv \in W_2$$

since W_1 and W_2 are closed under scalar multiplication. It then follows that

$$tv \in W_1 \cap W_2,$$

which shows that $W_1 \cap W_2$ is closed under scalar multiplication.

We have shown that $W_1 \cap W_2$ is a non–empty subset of $\mathbb{R}^2$ which is closed under the vector space operations of $\mathbb{R}^n$; that is, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

Applying part (a) in this problem we also conclude that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ which is contained in both W_1 and W_2. $\square$