Solutions to Assignment #5

1. Let S_1 and S_2 denote two subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.

   (a) Prove that $\text{span}(S_1) \subseteq \text{span}(S_2)$.

   Proof: Since $S_2 \subseteq \text{span}(S_2)$, it follows from $S_1 \subseteq S_2$ that

   $$S_1 \subseteq \text{span}(S_2).$$

   Thus, since $\text{span}(S_2)$ is a subspace and $\text{span}(S_1)$ is the smallest subspace of $\mathbb{R}^n$ which contains S_1, we have that

   $$\text{span}(S_1) \subseteq \text{span}(S_2),$$

   which was to be shown. □

   (b) Prove that if S_1 spans $\mathbb{R}^n$, then $\text{span}(S_2) = \mathbb{R}^n$.

   Proof: Since $\text{span}(S_1) = \mathbb{R}^n$, it follows from part (a) that

   $$\mathbb{R}^n \subseteq \text{span}(S_2).$$

   Moreover, $\text{span}(S_2) \subseteq \mathbb{R}^n$, since $\text{span}(S_2)$ is a subspace of $\mathbb{R}^n$. We therefore conclude that $\text{span}(S_2) = \mathbb{R}^n$. □

2. Let $S = \{v_1, v_2, \ldots v_k\}$, where be $v_1, v_2, \ldots v_k$ are vectors in $\mathbb{R}^n$. The symbol $S\backslash\{v_j\}$ denotes the set S with v_j removed from the set, for $j \in \{1, 2, \ldots, k\}$.

   Suppose that $v_j \in \text{span}(S\backslash\{v_j\})$ for some j in $\{1, 2, \ldots, k\}$. Prove that

   $$\text{span}(S\backslash\{v_j\}) = \text{span}(S).$$

   Proof: Observe that $S\backslash\{v_j\} \subseteq S$. Consequently, by part (a) in Problem 1,

   $$\text{span}(S\backslash\{v_j\}) \subseteq \text{span}(S).$$

   It remains to show, therefore, that

   $$\text{span}(S) \subseteq \text{span}(S\backslash\{v_j\}).$$

   To show this, let $v \in \text{span}(S)$, then

   $$v = c_1 v_1 + c_2 v_2 + \cdots + c_j v_j + \cdots + c_k v_k, \tag{1}$$

for some scalars $c_1, c_2, \ldots, c_k$. Now, since $v_j \in \text{ span}(S\backslash\{v_j\})$, there exist scalars $d_1, d_2, \ldots, d_{j-1}, d_{j+1}, \ldots, d_k$ such that

$$v_j = d_1 v_1 + d_2 v_2 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k.$$

Substituting for v_j in (1) and using the distributive propeties, we then get that

$$\begin{aligned} v &= c_1 v_1 + \cdots + c_j(d_1 v_1 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k) + \cdots + c_k v_k \\ &= (c_1 + c_j d_1) v_1 + (c_2 + c_j d_2) v_2 + \cdots + (c_{j-1} + c_j d_{j-1}) v_{j-1} \\ &\qquad + (c_{j+1} + c_j d_{j+1}) v_{j+1} + \cdots + (c_k + c_j d_k) v_k, \end{aligned}$$

which is a linear combination of vectors is $S\backslash\{v_j\}$. It then follows that

$$v \in \text{ span}(S) \Rightarrow v \in \text{ span}(S\backslash\{v_j\}),$$

or

$$\text{span}(S) \subseteq \text{ span}(S\backslash\{v_j\}),$$

which finishes the proof. □

3. Suppose that W is a subspace of $\mathbb{R}^n$ and that $v_1, v_2, \ldots, v_k \in W$. Prove that

$$\text{span}\{v_1, v_2, \ldots, v_k\} \subseteq W.$$

Proof: Put $S = \{v_1, v_2, \ldots, v_k\}$; then $S \subseteq W$, where W is a subspace of $\mathbb{R}^n$. It then follows that

$$\text{span}(S) \subseteq W,$$

since span(S) is the smallest subspace of $\mathbb{R}^n$ which contains S. □

4. Let W be a subspace of $\mathbb{R}^n$. Prove that if the set $\{v, w\}$ spans W, then the set $\{v, v + w\}$ also spans W.

Proof: Suppose that $W = \text{ span}\{v, w\}$. Then, W is a subspace which contains v and w. In then follows from the closure of W with respect to vector addition that $v + w \in W$. We then have that

$$v, v + w \in W.$$

Thus, by the result of Problem 3,

$$\text{span}\{v, v + w\} \subseteq W. \qquad (2)$$

Ont the other hand, since $W = \text{ span}\{v, w\}$, $u \in W$ implies that

$$u = c_1 v + c_2 w,$$

for some scalars c_1 and c_2. Consequently,

$$\begin{array}{rcl} u & = & c_1 v + c_2 w + c_2 v - c_2 v \\ & = & (c_1 - c_2) v + c_2 (w + v), \end{array}$$

which shows that $u \in \text{ span}\{v, v + w\}$; thus,

$$u \in W \Rightarrow u \in \text{ span}\{v, v + w\},$$

or

$$W \subseteq \text{ span}\{v, v + w\}.$$

Combining this with (2) yields that

$$W = \text{ span}\{v, v + w\};$$

that is, the set $\{v, v + w\}$ spans W. □

5. Let W be the solution set of the homogeneous system

$$\left\{ \begin{array}{rcl} -x_1 + 2x_2 - 3x_3 & = & 0 \\ 2x_1 - x_2 + 4x_3 & = & 0. \end{array} \right.$$

Solve the system to determine W, and find a set, S, of vectors in $\mathbb{R}^3$ such that

$$W = \text{ span}(S).$$

Deduce, therefore, that W is a subspace of $\mathbb{R}^3$.

***Solution**:* Solve the first equation for x_1 and substitute into the second equation to get that

$$\left\{ \begin{array}{lcl} -x_1 + 2x_2 - 3x_3 & = & 0 \\ 3x_2 - 2x_3 & = & 0. \end{array} \right. \tag{3}$$

Next, solve for x_2 in the second equation in system (3) and substitute into the first equation to get

$$\left\{ \begin{array}{lcl} -x_1 - \frac{5}{3}x_3 & = & 0 \\ 3x_2 - 2x_3 & = & 0. \end{array} \right. \tag{4}$$

Solving for x_1 and x_2 in system (4) then yields

$$\begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = \frac{2}{3}x_3. \end{cases} \tag{5}$$

Setting $x_3 = 3t$, where t is an arbitrary parameter, t, then gives the solutions

$$\begin{cases} x_1 = -5t \\ x_2 = 2t \\ x_3 = 3t. \end{cases} \tag{6}$$

We then get that the solution space for the system is

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix},\ t \in \mathbb{R} \right\},$$

or

$$W = \operatorname{span}(S),$$

where

$$S = \left\{ \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Since the span of any set is a subspace, it follows that W is a subspace of $\mathbb{R}^3$. □