Solutions to Assignment #7

1. Prove that if a homogeneous system of linear equations has one nontrivial solution, then it has infinitely many solutions.

   Proof: Let $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a nontrivial solution of (1):

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0, \end{array} \right. \qquad (1)$$

   Then, $x_1, x_2, \ldots, x_n$ are not all zero and they satisfy the equations in (1). Multiply the equations by a nonzero scalar, t, and apply the distributive and associative properties to get

$$\left\{ \begin{array}{rcl} a_{11}(tx_1) + a_{12}(tx_2) + \cdots + a_{1n}(tx_n) & = & 0 \\ a_{21}(tx_1) + a_{22}(tx_2) + \cdots + a_{2n}(tx_n) & = & 0 \\ \vdots & = & \vdots \\ a_{m1}(tx_1) + a_{m2}(tx_2) + \cdots + a_{mn}(tx_n) & = & 0. \end{array} \right. \qquad (2)$$

   It then follows from (2) that $\begin{pmatrix} tx_1 \\ tx_2 \\ \vdots \\ tx_n \end{pmatrix}$, where t is an arbitrary parameter, solves system (1). Hence, since the vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is not the zero vector, the homogeneous system in (1) has infinitely many solutions. □

2. Consider the vectors v_1, v_2, v_3 and v_4 in $\mathbb{R}^4$ given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \\ -5 \end{pmatrix}, \quad \text{and} \quad v_4 = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Determine whether the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent; if not, find a linearly independent subset of $\{v_1, v_2, v_3, v_4\}$ which spans $\operatorname{span}\{v_1, v_2, v_3, v_4\}$.

***Solution**:* Consider the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}. \tag{3}$$

This leads to the system

$$\left\{ \begin{array}{rcl} c_1 + 2c_2 + c_4 & = & 0 \\ -c_2 - c_3 - 3c_4 & = & 0 \\ -c_1 + c_2 + 3c_3 & = & 0 \\ 2c_1 - c_2 - 5c_3 + c_4 & = & 0. \end{array} \right. \tag{4}$$

We use Gaussian elimination to reduce the system

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 \\ -1 & 1 & 3 & 0 & 0 \\ 2 & -1 & -5 & 1 & 0 \end{array} \right).$$

We present below the resulting matrices with the elementary row operation indicated at each row:

$$\begin{array}{r} R_1 \\ R_2 \\ R_1 + R_3 \\ -2R_1 + R_4 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 \\ 0 & 3 & 3 & 1 & 0 \\ 0 & -5 & -5 & -1 & 0 \end{array} \right)$$

$$\begin{array}{r} R_1 \\ -R_2 \\ R_1 + R_3 \\ -2R_1 + R_4 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 3 & 3 & 1 & 0 \\ 0 & -5 & -5 & -1 & 0 \end{array} \right)$$

$$\begin{array}{r} R_1 \\ -R_2 \\ 3R_2 + R_1 + R_3 \\ -5R_2 - 2R_1 + R_4 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 14 & 0 \end{array} \right)$$

$$
\begin{array}{r}
R_1 \\
-R_2 \\
-\frac{3}{8}R_2 - \frac{1}{8}R_1 - \frac{1}{8}R_3 \\
-\frac{5}{14}R_2 - \frac{1}{7}R_1 + \frac{1}{14}R_4
\end{array}
\qquad
\left(\begin{array}{cccc|c}
1 & 2 & 0 & 1 & 0 \\
0 & 1 & 1 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

$$
\begin{array}{r}
R_1 \\
-R_2 \\
-\frac{3}{8}R_2 - \frac{1}{8}R_1 - \frac{1}{8}R_3 \\
\frac{1}{56}R_2 - \frac{1}{56}R_1 + \frac{1}{8} + R_3 + \frac{1}{14}R_4
\end{array}
\qquad
\left(\begin{array}{cccc|c}
1 & 2 & 0 & 1 & 0 \\
0 & 1 & 1 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

We then get that the system in (4) is equivalent to the system

$$
\left\{\begin{array}{rcl}
c_1 + 2c_2 + c_4 & = & 0 \\
c_2 + c_3 + 3c_4 & = & 0 \\
c_4 & = & 0,
\end{array}\right. \tag{5}
$$

which has more unknowns than equations. It then follows from the Fundamental Theorem of Homogeneous Linear Systems that system (4) has nontrivial solutions. Consequently, the vector equation in (3) has nontrivial solutions, and, therefore, the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

To find a linearly independent subset of $\{v_1, v_2, v_3, v_4\}$ which spans $\text{span}\{v_1, v_2, v_3, v_4\}$, we continue the Gauss–Jordan reduction process of the system in (5) to get to the system

$$
\left\{\begin{array}{rcl}
c_1 - 2c_3 & = & 0 \\
c_2 + c_3 & = & 0 \\
c_4 & = & 0,
\end{array}\right.
$$

We then get the solutions

$$
\left\{\begin{array}{rcl}
c_1 & = & 2t \\
c_2 & = & - = t \\
c_3 & = & t \\
c_4 & = & 0,
\end{array}\right.
$$

where t is arbitrary. Taking $t = 1$, we get the solution:

$$
c_1 = 2,\ c_2 = -1,\ c_3 = 1,\ c_4 = 0.
$$

This yields the vector relation

$$2v_1 - v_2 + v_3 = \mathbf{0}.$$

from the vector equation (3), which shows that $v_3 \in \text{span}\{v_1, v_2\}$. It then follows that

$$\text{span}\{v_1, v_2, v_4\} = \text{span}\{v_1, v_2, v_3, v_4\}.$$

To complete the solution, we need to show that $\{v_1, v_3, v_4\}$ is linearly independent. To do so, start with the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_4 = \mathbf{0}, \tag{6}$$

which leads to the system

$$\left\{ \begin{array}{rcl} c_1 + 2c_2 + c_3 & = & 0 \\ -c_2 - 3c_3 & = & 0 \\ -c_1 + c_2 & = & 0 \\ 2c_1 - c_2 + c_3 & = & 0. \end{array} \right. \tag{7}$$

Next, perform elementary row operations of the augmented matrix corresponding to the system in (7),

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{array} \right),$$

to get

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -5 & -1 & 0 \end{array} \right),$$

where we have performed: $R_1 + R_3 \to R_3$ and $-2R_1 + R_4 \to R_4$. Continue now performing: $-R_2 \to R_2$, $-3R_2 + R_3 \to R_3$ and $5R_2 + R_4 \to R_4$ in succession, to get

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 14 & 0 \end{array} \right).$$

Finally, perform $-\frac{1}{8}R_3 \to R_3$ and $-14R_3 + R_4 \to R_4$ in succession to get

$$\begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Consequently, the system in (7) is equivalent to the system

$$\begin{cases} c_1 + 2c_2 + c_3 &= 0 \\ c_2 + 3c_3 &= 0 \\ c_3 &= 0. \end{cases} \tag{8}$$

The system in (8) cab be seen to have only the trivial solution

$$c_1 = c_2 = c_3 = 0;$$

therefore, the system (7) has only the trivial solution. Consequently, the vector equation in (13) has only the trivial solution. Hence, the set $\{v_1, v_3, v_4\}$ is linearly independent. □

3. Let

$$W = \text{ span}\left(\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}\right\}\right).$$

Find a linearly independent subset of W which spans W.

Solution*:* Denote the vectors in the set

$$\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}\right\}$$

by v_1, v_2, v_3 and v_4, respectively, we look for a linear vector relation of the form

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}. \tag{9}$$

This leads to the system

$$\begin{cases} c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + 2c_2 + 2c_3 + 4c_4 &= 0 \\ c_1 + 2c_3 + 3c_4 &= 0. \end{cases} \tag{10}$$

The augmented matrix of this system is:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \quad \left(\begin{array}{cccc|c} 1 & 3 & -1 & 0 & 0 \\ 2 & 2 & 2 & 4 & 0 \\ 1 & 0 & 2 & 3 & 0 \end{array}\right).$$

We can reduce this matrix to

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & -3 & 3 & 3 & 0 \end{array}\right),$$

where we have performed the elementary row operations $\frac{1}{2}R_2 \to R_2$, $-R_1 + R_2 \to R_2$, and $-R_1 + R_3 \to R_3$ in succession. Next, perform the row operations: $-\frac{1}{2}R_2 \to R_2$ and $3R_2 + R_3 \to R_3$ in succession to obtain

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which is in row–echelon form. To get the reduced row–echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

perform the operation $-3R_3 + R_1 \to R_1$. This yields the system

$$\left\{ \begin{array}{rcl} c_1 + 2c_3 + 3c_4 & = & 0 \\ c_2 - c_3 - c_4 & = & 0, \end{array} \right. \tag{11}$$

which is equivalent to system (10). Solving for the leading variables in (11) yields the solutions

$$\left\{ \begin{array}{rcl} c_1 & = & 2t + 3s \\ c_2 & = & -t - s \\ c_3 & = & -t \\ c_4 & = & -s, \end{array} \right. \tag{12}$$

where t and s are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (12) yields from (9) the linear relation

$$2v_1 - v_2 - v_3 = \mathbf{0},$$

which shows that $v_3 = 2v_1 - v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.
Similarly, taking $t = 0$ and $s = 1$ in (12) yields

$$3v_1 - v_2 + v_4 = \mathbf{0},$$

which shows that $v_4 = -3v_1 + v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.
We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is $\{v_1, v_2\}$ spans W.
It remains to show that $\{v_1, v_2\}$ is linearly independent. To prove this, consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}, \tag{13}$$

which leads to the system

$$\left\{\begin{array}{rcl} c_1 + 2c_2 & = & 0 \\ -c_2 & = & 0 \\ -c_1 + c_2 & = & 0 \\ 2c_1 - c_2 & = & 0, \end{array}\right.$$

which can be seen to have only the trivial solution: $c_1 = c_2 = 0$. It then follows that the vector equation (13) has only the trivial solution, and therefore $\{v_1, v_2\}$ is linearly independent. □

4. Let W denote the solution space of the system

$$\left\{\begin{array}{lcl} 3x_1 - 2x_2 - 2x_3 - x_4 + x_5 & = & 0 \\ x_1 - 3x_2 - 2x_5 & = & 0 \\ 2x_2 + x_3 + 2x_4 - x_5 & = & 0 \\ -x_1 + x_2 - x_3 + x_4 - x_5 & = & 0. \end{array}\right.$$

Find a linearly independent subset, S, of $\mathbb{R}^5$ such that $W = \text{span}(S)$.

***Solution**:* Begin with the augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrrr|r} 3 & -2 & -2 & -1 & 1 & 0 \\ 1 & -3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 1 & -1 & 0 \end{array}\right),$$

and perform the elementary row operations: $R_1 \leftrightarrow R_2$, $-3R_1 + R_2 \to R_2$, and $R_1 + R_4 \to R_4$, in succession to get the matrix

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 0 & 0 & -2 & 0 \\ 0 & 7 & -2 & -1 & 7 & 0 \\ 0 & 2 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & 1 & -3 & 0 \end{array}\right),$$

Next, perform: $R_2 \leftrightarrow R_3$, $\frac{1}{2}R_2 \to R_2$, $-7R_2 + R_3 \to R_3$, and $2R_2 + R_4 \to R_4$ in succession to get

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -11/2 & -8 & 21/2 & 0 \\ 0 & 0 & 0 & 3 & -4 & 0 \end{array}\right).$$

Performing the elementary row operations: $-(2/11)R_3 \to R_3$ and $(1/3)R_4 \to R_4$ then yields the row–echelon form

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 16/11 & -21/11 & 0 \\ 0 & 0 & 0 & 1 & -4/3 & 0 \end{array}\right).$$

Finally, perform the elementary row operations: $(-16/11)R_4 + R_3 \to R_3$, $-R_4 + R_2 \to R_2$, $(-1/2)R_3 + R_2 \to R_2$, and $3R_2 + R_1 \to R_1$, to get the reduced row–echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 5/11 & 0 \\ 0 & 1 & 0 & 0 & 9/11 & 0 \\ 0 & 0 & 1 & 0 & 1/33 & 0 \\ 0 & 0 & 0 & 1 & -4/3 & 0 \end{array}\right).$$

We therefore obtain that W is the span of the vector $\begin{pmatrix} -5/11 \\ -9/11 \\ -1/33 \\ 4/3 \\ 1 \end{pmatrix}$, or any non–zero multiple of the vector. We therefore conclude that $W = \operatorname{span}(S)$, where

$$S = \left\{ \begin{pmatrix} -5/11 \\ -9/11 \\ -1/33 \\ 4/3 \\ 1 \end{pmatrix} \right\}.$$

□

5. Determine whether or not the vector $\begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}$ lies in the span of the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right\}.$$

***Solution**:* To answer this question, we need to determine whether the vector equation

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix} \tag{14}$$

has a solution or not.

Equation (14) leads to the non–homogeneous system

$$\left\{\begin{array}{lcl} c_1 + c_3 + c_4 & = & 4 \\ c_1 - c_2 + 2c_3 - c_4 & = & 7 \\ 3c_1 + 3c_3 + 3c_4 & = & 7 \\ c_2 + 3c_3 - 2c_4 & = & 4, \end{array}\right.$$

which in turn yields the augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrrcr} 1 & 0 & 1 & 1 & | & 4 \\ 1 & -1 & 2 & -1 & | & 7 \\ 3 & 0 & 3 & 3 & | & 7 \\ 0 & 1 & 3 & -2 & | & 4 \end{array}\right).$$

We reduce this matrix by performing the elementary row operations: $-R_1 + R_2 \to R_2$, $-3R_1 + R_3 \to R_3$ to get

$$\left(\begin{array}{rrrrcr} 1 & 0 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -2 & | & 3 \\ 0 & 0 & 0 & 0 & | & -5 \\ 0 & 1 & 3 & -2 & | & 4 \end{array}\right).$$

The third row in the previous matrix yields $0 = -5$, which is impossible. Therefore, the vector equation in (14) is not solvable. Hence, $\begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}$ is not in the span of the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right\}.$$

$\square$