Review Problems for Exam 1

1. Give a basis for the span of the following set of vectors in $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

2. Find a basis for the solution space of the system

$$\left\{ \begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & - & 2x_4 & = & 0 \\ -x_1 & & & + & x_3 & + & x_4 & = & 0, \end{array} \right.$$

   and compute its dimension.

3. Prove that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

4. Let v and w denote vectors in $\mathbb{R}^n$.

   (a) Show that if the set $\{v, w\}$ is a linearly independent subset of $\mathbb{R}^n$ if and only if the set $\{v + w, v - w\}$ is linearly independent.

   (b) Show that $\text{span}\{v, w\} = \text{span}\{v + w, v - w\}$.

5. Let $\{u, v, w\}$ be a linearly independent subset of $\mathbb{R}^n$. Show that the set

$$\{u + v, u + w, v + w\}$$

   is linearly independent.

6. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Suppose there exists $v \in \mathbb{R}^n$ such that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

7. Let S denote a nonempty subset of $\mathbb{R}^n$. Assume that there exists $v \in S$ such that $v \in \text{span}(S \backslash \{v\})$. Show that $\text{span}(S \backslash \{v\}) = \text{span}(S)$.

8. Let S_1 and S_2 be subsets of $\mathbb{R}^n$. Suppose that $S_1 \cup S_2$ is linearly independent and that $S_1 \cap S_2 = \emptyset$. Show that $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$.

9. Let J and H be planes in $\mathbb{R}^3$ given by

$$J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 2y + z = 0 \right\}.$$

   (a) Give bases for J and H and compute their dimensions.

   (b) Give a basis for the subspace $J \cap H$ and compute $\dim(J \cap H)$.

10. Let W be a subspace of $\mathbb{R}^n$.

   (a) Prove that if $v \in W$ and $v \neq \mathbf{0}$, then $rv = sv$ implies that $r = s$, where r and s are scalars.

   (b) Prove that if W has more than one element, then W has infinitely many elements.

11. Let W be a subspace of $\mathbb{R}^n$ and S_1 and S_2 be subsets of W.

   (a) Show that $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

   (b) Give an example in which $\operatorname{span}(S_1 \cap S_2) \neq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

12. Let W be a subspace of $\mathbb{R}^n$ of dimension k, where $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ denote a basis for W.

   Show that there exist vectors $v_1, v_2, \ldots, v_\ell$ in $\mathbb{R}^n$ such that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell\}$$

   is a basis for $\mathbb{R}^n$. What is ℓ in terms of n and k?

13. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. We write $W_1 \oplus W_2$ for the subspace $W_1 + W_2$ for the special case in which $W_1 \cap W_2 = \{\mathbf{0}\}$. Show that every vector $v \in W_1 \oplus W_2$ can be written in the form $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, in one and only one way; that is, if $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$, then $u_1 = v_1$ and $u_2 = v_2$.

14. Let W be a k–dimensional subspace of $\mathbb{R}^n$, and let $\{v_1, v_2, \ldots, v_k\}$ be a subset of W.

   (a) Show that if $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, then it must span W.

   (b) Show that if $\{v_1, v_2, \ldots, v_k\}$ span W, then it is linearly independent.

15. Let A denote the $n \times k$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix},$$

and denote the columns of A by $w_1, w_2, \ldots, w_k$, respectively.

(a) Show that the set $\{w_1, w_2, \ldots, w_k\}$ is a linearly independent subset of $\mathbb{R}^n$ if and only if the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k & = & 0 \end{cases}$$

has only the trivial solution.

(b) Let $v = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ be any vector in $\mathbb{R}^n$.

Show that $v \in \text{span}(\{w_1, w_2, \ldots, w_k\})$ if and only if the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k & = & b_n \end{cases}$$

has a solution.