Solutions to Review Problems for Exam 1

1. Give a basis for the span of the following set of vectors in $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

***Solution**:* Denote the vectors in the set

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}$$

by v_1, v_2, v_3 and v_4, respectively. We look for a linear vector relation of the form

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}. \tag{1}$$

This leads to the system

$$\left\{ \begin{array}{rcl} c_1 - 2c_2 + c_3 + c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0 \\ c_1 + 3c_2 + 6c_3 - 4c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0. \end{array} \right. \tag{2}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrrcr} 1 & -2 & 1 & 1 & | & 0 \\ -1 & 0 & -3 & 1 & | & 0 \\ 1 & 3 & 6 & -4 & | & 0 \\ -1 & 0 & -3 & 1 & | & 0 \end{array} \right).$$

We can reduce this matrix to

$$\left(\begin{array}{rrrrcr} 1 & 0 & 3 & -1 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{array} \right),$$

which is in reduced row–echelon form. We therefore get that the system in (2) is equivalent to the system

$$\left\{ \begin{array}{rcl} c_1 + 3c_3 - c_4 & = & 0 \\ c_2 + c_3 - c_4 & = & 0. \end{array} \right. \tag{3}$$

Solving for the leading variables in (3) yields the solutions

$$\begin{cases} c_1 &= 3t + s \\ c_2 &= t + s \\ c_3 &= -t \\ c_4 &= s, \end{cases} \qquad (4)$$

where t and s are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (4) yields from (1) the linear relation

$$3v_1 + v_2 - v_3 = \mathbf{0},$$

which shows that $v_3 = -3v_1 - v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.
Similarly, taking $t = 0$ and $s = 1$ in (4) yields

$$v_1 + v_2 + v_4 = \mathbf{0},$$

which shows that $v_4 = -v_1 - v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.
We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is, $\{v_1, v_2\}$ spans $\text{span}\{v_1, v_2, v_3, v_4\}$.
To see that $\{v_1, v_2\}$ is linearly independent, observe that v_1 and v_2 are not multiples of each other. We therefore conclude that $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$. □

2. Find a basis for the solution space of the system

$$\begin{cases} x_1 - x_2 + x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_4 &= 0 \\ -x_1 + x_3 + x_4 &= 0, \end{cases} \qquad (5)$$

and compute its dimension.

***Solution**:* We first find the solution space, W, of the system. In order to do this, we reduce the augmented matrix of this system,

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \qquad \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 2 & -1 & 0 & -2 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right),$$

to its reduced row–echelon form:

$$\left(\begin{array}{rrrr|r} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Consequently, the system in (5) is equivalent to the system

$$\left\{ \begin{array}{rcrcrcl} x_1 & - & x_3 & - & x_4 & = & 0 \\ & x_2 - & 2x_3 & & & = & 0. \end{array} \right. \tag{6}$$

Solving for the leading variables in the system in (6) we obtain the solutions

$$\left\{ \begin{array}{rcl} x_1 & = & t+s \\ x_2 & = & 2t \\ x_3 & = & t \\ x_4 & = & s, \end{array} \right.$$

where t and s are arbitrary parameters. I then follows that the solution space of system (6) is

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for W and therefore $\dim(W) = 2$. □

3. Prove that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

Proof: Let v_1 , v_2, v_3 and v_4 denote four vectors in $\mathbb{R}^3$ and write

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}.$$

Consider the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4, = \mathbf{0}. \tag{7}$$

This equation translates into the homogeneous system

$$\left\{ \begin{array}{rcl} a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + a_{14}c_4 & = & 0 \\ a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + a_{24}c_4 & = & 0 \\ a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + a_{34}c_4 & = & 0, \end{array} \right. \tag{8}$$

of 3 linear equations in 4 unknowns. It then follows from the Fundamental Theorem for Homogeneous Linear Systems that system (8) has infinitely many solutions. Consequently, the vector equation in (7) has a nontrivial solution, and therefore the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. □

4. Let v and w denote vectors in $\mathbb{R}^n$.

   (a) Show that if the set $\{v, w\}$ is a linearly independent subset of $\mathbb{R}^n$ if and only if the set $\{v + w, v - w\}$ is linearly independent.

   (b) Show that $\text{span}\{v, w\} = \text{span}\{v + w, v - w\}$.

   Solution:

   (a) First we prove that if $\{v, w\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v + w, v - w\}$.

   Proof: Assume that $\{v, w\}$ is a linearly independent and consider the vector equation

$$c_1(v + w) + c_2(v - w) = \mathbf{0}. \tag{9}$$

   Applying the distributive and associative properties, the equation in (9) turns into

$$(c_1 + c_2)v + (c_1 - c_2)w = \mathbf{0}. \tag{10}$$

   It follows from (10) and the linear independence of $\{v, w\}$ that

$$\left\{ \begin{array}{rcl} c_1 + c_2 & = & 0 \\ c_1 - c_2 & = & 0. \end{array} \right. \tag{11}$$

The system in (11) has only the trivial solution: $c_2 = c_1 = 0$. Hence, the vector equation in (9) has only the trivial solution and therefore the set $\{v + w, v - w\}$ is linearly independent. $\square$

Next, we prove the converse: If $\{v + w, v - w\}$ is linearly independent, then $\{v, w\}$ is a linearly independent.

Proof: Assume that $\{v + w, v - w\}$ is a linearly independent and assume, by way of contradiction, that the set $\{v, w\}$ is linearly dependent. It then follows that

$$w = cv, \tag{12}$$

for some scalar c.
We first see that the scalar, c, in (12) cannot be 1; for, if $c = 1$, $v - w = \mathbf{0}$, and $\{v + w, v - w\}$ would be linearly dependent, which contradicts the assumption of linear independence of $\{v + w, v - w\}$. We then have that $c \neq 1$ in (12).
It follows from (12) that

$$v + w = (1 + c)v \tag{13}$$

and

$$v - w = (1 - c)v. \tag{14}$$

Rewrite (13) as

$$v + w = \frac{1 + c}{1 - c}(1 - c)v,$$

and use (14) to get that

$$v + w = \frac{1 + c}{1 - c}(v - w),$$

which shows that the set $\{v+w, v-w\}$ is linearly dependent. This is a contradiction. Hence, $\{v, w\}$ is linearly independent. $\square$

(b) In order to prove that $\text{span}\{v, w\} = \text{span}\{v + w, v - w\}$, we establish the following inclusions:

(i) $\text{span}\{v, w\} \subseteq \text{span}\{v + w, v - w\}$, and

(ii) $\text{span}\{v + w, v - w\} \subseteq \text{span}\{v, w\}$.

Proof of (i): First observe that

$$(v+w)+(v-w)=2v,$$

so that

$$v=\frac{1}{2}(v+w)+\frac{1}{2}(v-w);$$

consequently,

$$v\in \text{span}\{v+w, v-w\}. \tag{15}$$

Similarly, since

$$w=\frac{1}{2}(v+w)-\frac{1}{2}(v-w),$$

it follows that

$$w\in \text{span}\{v+w, v-w\}. \tag{16}$$

Combining (15) and (16) we see that

$$\{v,w\}\subseteq \text{span}\{v+w, v-w\},$$

which implies that

$$\text{span}\{v,w\}\subseteq \text{span}\{v+w, v-w\},$$

since $\text{span}\{v,w\}$ is the smallest subspace of $\mathbb{R}^n$ that contains the set $\{v,w\}$. We have therefore established (i). □

Proof of (ii): Note that

$$v+w\in \text{span}\{v,w\} \quad \text{and} \quad v-w\in \text{span}\{v,w\},$$

so that

$$\{v+w, v-w\}\subseteq \text{span}\{v,w\}.$$

Hence,

$$\text{span}\{v+w, v-w\}\subseteq \text{span}\{v,w\},$$

since $\text{span}\{v+w, v-w\}$ is the smallest subspace of $\mathbb{R}^n$ that contains $\{v+w, v-w\}$. This concludes the proof of (ii). □

□

5. Let $\{u, v, w\}$ be a linearly independent subset of $\mathbb{R}^n$. Show that the set

$$\{u + v, u + w, v + w\}$$

is linearly independent.

Solution: Assume that $\{u, v, w\}$ be a linearly independent and consider the vector equation

$$c_1(u + v) + c_2(u + w) + c_3(v + w) = \mathbf{0}. \tag{17}$$

Next, use the distributive, associative and commutative properties of the vector space operations to rewrite (17) as

$$(c_1 + c_2)u + (c_1 + c_3)v + (c_2 + c_3)w = \mathbf{0}. \tag{18}$$

It follows from (18) and the linear independence of the set $\{u, v, w\}$ that

$$\left\{\begin{array}{rcl} c_1 + c_2 & = & 0 \\ c_1 + c_3 & = & 0 \\ c_2 + c_3 & = & 0. \end{array}\right. \tag{19}$$

To solve the system in (19), use Gaussian eliminations on the augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}$$

to obtain

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

It then follows that the system (19) has only the trivial solution

$$c_1 = c_2 = c_2 = 0,$$

which implies that the set

$$\{u + v, u + w, v + w\}$$

is linearly independent. $\square$

6. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Suppose there exists $v \in \mathbb{R}^n$ such that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

   Proof: Assume that $S = \{v_1, v_2, \ldots, v_k\}$ is a linearly independent subset of $\mathbb{R}^n$ and that $v \in \mathbb{R}^n$ is such that $v \notin \text{span}(S)$. Suppose that

   $$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + cv = 0. \tag{20}$$

   We first see that c in (20) must be 0; otherwise, $c \neq 0$ and we can solve for v in (20) to get that

   $$v = -\frac{c_1}{c} v_1 - \frac{c_2}{c} v_2 - \cdots - \frac{c_k}{c} v_k,$$

   which shows that $v \in \text{span}(S)$; this contradicts the assumption that $v \notin \text{span}(S)$. Hence, $c = 0$ and so we obtain from (20) that

   $$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0. \tag{21}$$

   It follows from (21) and the assumption that S is linearly independent that

   $$c_1 = c_2 = \ldots = c_k = 0.$$

   We have therefore shown that (20) implies that

   $$c_1 = c_2 = \ldots = c_k = c = 0.$$

   Hence, the set $S \cup \{v\}$ is linearly independent. □

7. Let S denote a nonempty subset of $\mathbb{R}^n$. Assume that there exists $v \in S$ such that $v \in \text{span}(S\backslash\{v\})$. Show that

   $$\text{span}(S\backslash\{v\}) = \text{span}(S).$$

   Proof: Let $S \subseteq \mathbb{R}^n$ and assume that there exists $v \in S$ such that $v \in \text{span}(S\backslash\{v\})$. First observe that $S\backslash\{v\} \subseteq S$, so that

   $$S\backslash\{v\} \subseteq \text{span}(S).$$

   Thus,

   $$\text{span}(S\backslash\{v\}) \subseteq \text{span}(S) \tag{22}$$

   because $\text{span}(S\backslash\{v\})$ is the smallest subspace of $\mathbb{R}^n$ that contains $S\backslash\{v\}$.

Next, let $w \in \text{span}(S)$. Then,

$$w = c_1 w_2 + c_2 w_2 + \cdots + c_k w_k + cv, \tag{23}$$

where

$$w_i \in S\backslash\{v\}, \quad \text{for } i = 1, 2, \ldots, k. \tag{24}$$

Next, use the assumption that $v \in \text{span}(S\backslash\{v\})$ to write

$$v = d_1 v_2 + d_2 v_2 + \cdots + d_\ell v_\ell, \tag{25}$$

where

$$v_j \in S\backslash\{v\}, \quad \text{for } j = 1, 2, \ldots, \ell. \tag{26}$$

It then follows from (23) and (25) that

$$w = c_1 w_2 + c_2 w_2 + \cdots + c_k w_k + c(d_1 v_2 + d_2 v_2 + \cdots + d_\ell v_\ell),$$

or

$$w = c_1 w_2 + c_2 w_2 + \cdots + c_k w_k + cd_1 v_2 + cd_2 v_2 + \cdots + cd_\ell v_\ell. \tag{27}$$

Consequently, in view of (24) and (26), we obtain from (27) that

$$w \in \text{span}(S\backslash\{v\}).$$

We have therefore shown that

$$\text{span}(S) \subseteq \text{span}(S\backslash\{v\}). \tag{28}$$

Combining (22) and (28) yields what we were asked to prove. □

8. Let S_1 and S_2 be subsets of $\mathbb{R}^n$. Suppose that $S_1 \cup S_2$ is linearly independent and that $S_1 \cap S_2 = \emptyset$. Show that $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$.

   ***Solution**:* Assume that $S_1 \cap S_2$ is linearly independent and that $S_1 \cap S_2 = \emptyset$.
   Let $v \in \text{span}(S_1) \cap \text{span}(S_2)$; then,

   $$v \in \text{span}(S_1) \quad \text{and} \quad v \in \text{span}(S_1).$$

   Thus, there exist $w_1, w_2, \ldots, w_k$ in S_1 and $v_1, v_2, \ldots, v_\ell$ in S_2 such that

   $$v = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k \tag{29}$$

and

$$v = d_1 v_1 + d_2 v_2 + \cdots + d_\ell v_\ell, \tag{30}$$

for scalars $c_1, c_2, \ldots, c_k$ and $d_1, d_2, \ldots, d_\ell$. It follows from (29) and (30) that

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = d_1 v_1 + d_2 v_2 + \cdots + d_\ell v_\ell,$$

from which we get

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k + (-d_1) v_1 + (-d_2) v_2 + \cdots + (-d_\ell) v_\ell = \mathbf{0}, \tag{31}$$

where

$$w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell \in S_1 \cup S_2.$$

It then follows from (31) and the assumptions that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ is linearly independent that

$$c_1 = c_2 = \cdots = c_k = d_1 = d_2 = \cdots = d_\ell = 0. \tag{32}$$

It then follows from (29) (or (30) and (32) that $v = \mathbf{0}$. Hence, $\text{span}(S_1) \cap \text{span}(S_2) = \{\mathbf{0}\}$. □

9. Let J and H be planes in $\mathbb{R}^3$ given by

$$J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\Big|\; 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\Big|\; x - 2y + z = 0 \right\}.$$

(a) Give bases for J and H and compute their dimensions.

Solution: To find a basis for J, we solve the equation

$$2x + 3y - 6z = 0$$

to get the solution space $J = \text{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. Thus, the set

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is a basis for J and so $\dim(J) = 2$.

Similarly, for H, we solve

$$x - 2y + z = 0$$

and obtain that

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a basis for H; thus, $\dim(H) = 2$. □

(b) Give a basis for the subspace $J \cap H$ and compute $\dim(J \cap H)$.

Solution: A vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in the intersection of J and H if it is a solution to the system of equations

$$\left\{ \begin{array}{rcl} 2x + 3y - 6z & = & 0 \\ x - 2y + z & = & 0. \end{array} \right. \tag{33}$$

Thus, to find $J \cap H$, we may use elementary row operations on the augmented matrix

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \quad \left(\begin{array}{rrr|r} 2 & 3 & -6 & 0 \\ 1 & -2 & 1 & 0 \end{array} \right)$$

to obtain the reduced matrix

$$\left(\begin{array}{rrr|r} 1 & 0 & -9/7 & 0 \\ 0 & 1 & -8/7 & 0 \end{array} \right).$$

Thus, the system in (33) is equivalent to

$$\left\{ \begin{array}{rcl} x - \frac{9}{7}z & = & 0 \\ y - \frac{8}{7}z & = & 0, \end{array} \right. \tag{34}$$

Solving for the leading variables in system (34) and setting $z = 7t$, where t is an arbitrary parameter, wee obtain that

$$J \cap H = \operatorname{span} \left\{ \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \right\}.$$

Thus, the set

$$\left\{ \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \right\}$$

is a basis for $J \cap H$ and, therefore, $\dim(J \cap H) = 1$. □

10. Let W be a subspace of $\mathbb{R}^n$.

    (a) Prove that if $v \in W$ and $v \neq \mathbf{0}$, then $rv = sv$ implies that $r = s$, where r and s are scalars.

    Proof: Suppose that $v \in W$, where W is a subspace of $\mathbb{R}^n$, and that $v \neq \mathbf{0}$. Suppose also that

    $$rv = sv \tag{35}$$

    for some scalars r and s. Add $-sv$ on both sides of the vector equation in (35) and apply the distributive property to obtain

    $$(r - s)v = \mathbf{0}. \tag{36}$$

    It follows from (36) and the assumption $v \neq \mathbf{0}$, that

    $$r - s = 0$$

    and therefore $r = s$, which was to be shown. □

    (b) Prove that if W has more than one element, then W has infinitely many elements.

    Proof: Since W has at least two elements, there has to be a vector, v, in W such that $v \neq \mathbf{0}$. Now, for any $t \in \mathbb{R}$, $tv \in W$ because W is closed under scalar multiplication. By part (a), $t_1 v \neq t_2 v$ for any $t_1 \neq t_2$. Consequently, W contains infinitely many vectors. □

11. Let W be a subspace of $\mathbb{R}^n$ and S_1 and S_2 be subsets of W.

    (a) Show that $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

    Proof: First observe that $S_1 \cap S_2 \subseteq S_1$ and $S_1 \cap S_2 \subseteq S_2$. Consequently,

    $$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \quad \text{and} \quad \operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_2).$$

    It then follows that

    $$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2),$$

    which was to be shown. □

    (b) Give an example in which $\operatorname{span}(S_1 \cap S_2) \neq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

Solution: Let $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$. Then, $S_1 \cap S_2 = \emptyset$ so that $\text{span}(S_1 \cap S_2) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^2$.
On the other hand,

$$\text{span}(S_1) = \text{span}(S_2)$$

because $\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence,

$$\text{span}(S_1) \cap \text{span}(S_2) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t \in \mathbb{R} \right\} \neq \{\mathbf{0}\}.$$

□

12. Let W be a subspace of $\mathbb{R}^n$ of dimension k, where $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ denote a basis for W.

Show that there exist vectors $v_1, v_2, \ldots, v_\ell$ in $\mathbb{R}^n$ such that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell\}$$

is a basis for $\mathbb{R}^n$. What is ℓ in terms of n and k?

Proof: Let W be a subspace of $\mathbb{R}^n$ and let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W. Assume that $k < n$. Then, $\text{span}(\{w_1, w_2, \ldots, w_k\}) = \mathbb{R}^n$; otherwise $\{w_1, w_2, \ldots, w_k\}$ would be a basis for $\mathbb{R}^n$, and therefore $\dim(\mathbb{R}^n) = k$, which is impossible since we are assuming that $k < n$. Thus, there exists $v_1 \in \mathbb{R}^n$ such that $v_2 \notin \text{span}(\{w_1, w_2, \ldots, w_k\})$. It then follows from the result of Problem 6 in this review sheet that the set

$$\{w_1, w_2, \ldots, w_k, v_1\}$$

is linearly independent.

We consider two possibilities: Either (i) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) = \mathbb{R}^n$, or (ii) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) \neq \mathbb{R}^n$.

If $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) = \mathbb{R}^n$, then $\{w_1, w_2, \ldots, w_k, v_1\}$ is a basis for $\mathbb{R}^n$ and $n = k + 1$. If not, there exists $v_2 \in \mathbb{R}^n$ such that

$$v_2 \notin \text{span}(\{w_1, w_2, \ldots, w_k, v_1\}).$$

It then follows from the result of Problem 6 that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2\}$$

is linearly independent.

Again, we consider two cases: Either (i) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) = \mathbb{R}^n$, or (ii) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) \neq \mathbb{R}^n$.

If $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) = \mathbb{R}^n$, then $\{w_1, w_2, \ldots, w_k, v_1, v_2\}$ is a basis for $\mathbb{R}^n$ and $n = k + 2$. If not, there exists $v_3 \in \mathbb{R}^n$ such that

$$v_3 \notin \text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}).$$

We continue in this fashion until we get vectors $v_1, v_2, \ldots, v_\ell$ in $\mathbb{R}^n$ such that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell\} \text{ is linearly independent} \tag{37}$$

and

$$\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell\}) = \mathbb{R}^n. \tag{38}$$

It follows from (37) and (38) that $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_\ell\}$ is a basis for $\mathbb{R}^n$ and therefore $k + \ell = n$, from which we get that $\ell = n - k$. □

13. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. We write $W_1 \oplus W_2$ for the subspace $W_1 + W_2$ for the special case in which $V = W_1 \cap W_2 = \{\mathbf{0}\}$. Show that every vector $v \in W_1 \oplus W_2$ can be written in the form $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, in one and only one way; that is, if $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$, then $u_1 = v_1$ and $u_2 = v_2$.

    Proof: Suppose that W_1 and W_2 are two subspaces of $\mathbb{R}^n$ which have only the zero vector in common; that is, $W_1 \cap W_2 = \{\mathbf{0}\}$. Let v be any vector in $W_1 + W_2$. Then, $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. Suppose that v can also be written as $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$. Then,

    $$v_1 + v_2 = u_1 + u_2,$$

    from which we get that

    $$v_1 - u_1 = v_2 - u_2, \tag{39}$$

    where $v_1 - u_1 \in W_1$ and $v_2 - u_2 \in W_2$ since W_1 and W_2 are subspaces of $\mathbb{R}^n$. It also follows from (39) that $v_1 - u_1 \in W_2$. Thus, $v_1 - u_1 \in W_1 \cap W_2 = \{\mathbf{0}\}$, which implies that

    $$v_1 - u_1 = \mathbf{0},$$

or

$$v_1 = u_1.$$

Similarly, we get that $v_2 = u_2$. □

14. Let W be a k–dimensional subspace of $\mathbb{R}^n$, and let $\{v_1, v_2, \ldots, v_k\}$ be a subset of W.

    (a) Show that if $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, then it must span W.

    Proof: Assume that W is a k–dimensional subspace of $\mathbb{R}^n$ and let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W.

    Suppose that $\{v_1, v_2, \ldots, v_k\}$ is linearly a independent subset of W. We show that $\{v_1, v_2, \ldots, v_k\}$ spans W.

    Arguing by contradiction, suppose that $\text{span}(\{v_1, v_2, \ldots, v_k\}) \neq W$. Then, there exists $v \in W$ such that $v \notin \text{span}(\{v_1, v_2, \ldots, v_k\})$. It then follows by the result of Problem 6 that the set

    $$\{v_1, v_2, \ldots, v_k, v\}$$

    is linearly a independent subset of W. However, since $\{v_1, v_2, \ldots, v_k, v\}$ has $k + 1$ elements and W has dimension k, $\{v_1, v_2, \ldots, v_k, v\}$ must be linearly dependent. We have therefore arrived at a contradiction. Hence, $\{v_1, v_2, \ldots, v_k\}$ must also span W. □

    (b) Show that if $\{v_1, v_2, \ldots, v_k\}$ span W, then it is linearly independent.

    Proof: Assume that W is a k–dimensional subspace of $\mathbb{R}^n$ and let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W.

    Suppose that $\text{span}(\{v_1, v_2, \ldots, v_k\}) = W$. We show that $\{v_1, v_2, \ldots, v_k\}$ is linearly a independent.

    Arguing by contradiction, suppose that $\{v_1, v_2, \ldots, v_k\}$ is linearly a dependent. Then, by reordering the vectors if necessary, we may assume that $v_k \in \text{span}(\{v_1, v_2, \ldots, v_{k-1}\})$. It then follows by the result of Problem 7 in this review sheet that

    $$\text{span}(\{v_1, v_2, \ldots, v_{k-1}\}) = \text{span}(\{v_1, v_2, \ldots, v_k\}) = W.$$

    Either $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly independent, or not. If $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly dependent, we may proceed as above to conclude that

    $$v_{k-1} \in \text{span}(\{v_1, v_2, \ldots, v_{k-2}\}),$$

(where, if necessary, the vectors have been rearranged); so that,

$$\text{span}(\{v_1, v_2, \ldots, v_{k-2}\}) = \text{span}(\{v_1, v_2, \ldots, v_{k-1}\}) = W.$$

Proceeding in this fashion we get to a subset of $\{v_1, v_2, \ldots, v_k\}$; namely, $\{v_1, v_2, \ldots, v_\ell\}$, where $\ell < k$, that is linearly independent and also spans W. In other words, $\{v_1, v_2, \ldots, v_\ell\}$ is a basis for W. Hence, $\dim(W) = \ell < k = \dim(W)$; this is a contradiction. Therefore, $\{v_1, v_2, \ldots, v_k\}$ must be linearly a independent. □

15. Let A denote the $n \times k$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix},$$

and denote the columns of A by $w_1, w_2, \ldots, w_k$, respectively.

(a) Show that the set $\{w_1, w_2, \ldots, w_k\}$ is a linearly independent subset of $\mathbb{R}^n$ if and only if the homogeneous system

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k & = & 0 \end{array}\right. \tag{40}$$

has only the trivial solution.

Proof: Note that $(c_1, c_2, \ldots, c_k)$ is a solution to the vector equation

$$c_1w_1 + c_2w_2 + \cdots + c_kw_k = \mathbf{0}. \tag{41}$$

if and only if

$$\left\{\begin{array}{rcl} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k & = & 0 \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2k}c_k & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nk}c_k & = & 0. \end{array}\right.$$

Hence, $(c_1, c_2, \ldots, c_k)$ is a solution of the vector equation in (41) if and only if $(c_1, c_2, \ldots, c_k)$ is a solution of the system in (40). Consequently, the vector equation in (41) has only the trivial solution if and only if the system (40) has only the trivial solution. Therefore, the set $\{w_1, w_2, \ldots, w_k\}$ is a linearly independent if and only of the system in (40) has only the trivial solution. $\square$

(b) Let $v = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ be any vector in $\mathbb{R}^n$.

Show that $v \in \text{span}(\{w_1, w_2, \ldots, w_k\})$ if and only if the system of linear equations

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k & = & b_n \end{array} \right. \tag{42}$$

has a solution.

Proof: $v \in \text{span}(\{w_1, w_2, \ldots, w_k\})$ if and only if

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + c_k \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix},$$

for some scalars $c_1, c_2, \ldots, c_k$. Thus, $v \in \text{span}(\{w_1, w_2, \ldots, w_k\})$ if and only if there exist scalars $c_1, 2_2, \ldots, c_k$ such that

$$\left\{ \begin{array}{rcl} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k & = & b_1 \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2k}c_k & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nk}c_k & = & b_n \end{array} \right.$$

Hence, $v \in \text{span}(\{w_1, w_2, \ldots, w_k\})$ if and only if the system in (42) has a solution. $\square$