Solutions to Exam 1

1. Answer the following questions as thoroughly as possible.

   (a) Let S denote a nonempty subset of $\mathbb{R}^n$. Give a precise definition of span(S).

   Answer: span(S) is the set of all finite linear combinations of vectors in S. □

   Alternative Answer: span(S) is the smallest subspace of $\mathbb{R}^n$ that contains S. □

   (b) Let W denote a subset of $\mathbb{R}^n$. State precisely what it means for W to be a subspace of $\mathbb{R}^n$.

   Answer: W is a subspace of $\mathbb{R}^n$ if and only if W is nonempty and W is closed under the vector space operations in $\mathbb{R}^n$. □

   Alternative Answer: W is a subspace of $\mathbb{R}^n$ if and only if W is a vector space with respect to the vector space operations in $\mathbb{R}^n$. □

   (c) Let W denote a subspace of $\mathbb{R}^n$. Give the definition of dim(W).

   Answer: dim(W) is the number of vectors in any basis for W. □

   (d) State the Fundamental Theorem for Homogeneous Linear Systems.

   Answer: A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions. □

2. Let S denote a subset of $\mathbb{R}^n$.

   (a) State precisely what it means for S to be linearly independent.

   Answer: S is linearly independent means that not vector in S is in the span of the other vectors in S. □

   Alternative Answer: S is linearly independent means that not vector in S is a linear combination of the other vectors in S. □

   (b) Let v and w denote vectors in $\mathbb{R}^n$. Prove that if the set $\{v, w\}$ is linearly independent, then the set $\{v, v + w\}$ is linearly independent.

   Proof: Assume that $\{v, w\}$ is linearly independent and consider the vector equation

   $$c_1 v + c_2(v + w) = \mathbf{0}, \tag{1}$$

   which may be rewritten as

   $$(c_1 + c_2)v + c_2 w = \mathbf{0}. \tag{2}$$

It follows from (2) and the assumption that the set $\{v, w\}$ is linearly independent that

$$\begin{cases} c_1 + c_2 = 0 \\ c_2 = 0. \end{cases} \tag{3}$$

The system in (3) can be solved to yield

$$c_1 = c_2 = 0.$$

Hence, the vector equation in (1) has only the trivial solution. Consequently, the set $\{v, v + w\}$ is linearly independent. $\square$

3. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - 2y + z = 0 \right\}$

   (a) Explain why W is a subspace of $\mathbb{R}^3$.

   ***Solution**:* W is the solution set of a linear, homogenous equation; hence, W is a subspace. $\square$

   ***Alternative Solution**:* Observe that $W = \text{span}\left(\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \right)$.

   Hence, W is a subspace of $\mathbb{R}^3$. $\square$

   (b) Verify that the set $B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a basis for W.

   ***Solution**:* First, see that the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are in W. Indeed, $2 - 2(1) - 0 = 0$ and $1 - 2(0) + (-1) = 0$; thus, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in W$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in W$.

   Next, observe that the set B is linearly independent since the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are not multiples of each other. Thus, since W is a two dimensional subspace of $\mathbb{R}^3$, it follows that B is a basis for W. $\square$

(c) Let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Verify that $v \in W$ and give the coordinates of v relative to B; that is, compute $[v]_B$.

Solution: First, observe that $1 - 2(1) + 1 = 0$; consequently, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in W$.

Next, we find scalars c_1 and c_2 such that

$$c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};$$

that is, we find solutions to the system of

$$\begin{cases} 2c_1 + c_1 &= 1 \\ c_1 &= 1 \\ -c_2 &= 1, \end{cases}$$

which has solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We then have that

$$[v]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

□

4. Find a basis for the solution space, W, of the homogenous system

$$\begin{cases} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0, \end{cases} \tag{4}$$

and compute $\dim(W)$.

Solution: Solve for the leading variables in (4) to get

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3. \end{aligned}$$

Then, set $x_3 = t$, where t is an arbitrary parameter to get

$$\begin{aligned} x_1 &= 2t \\ x_2 &= -t \\ x_3 &= t. \end{aligned}$$

Thus, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W$ if and only if

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ -t \\ t \end{pmatrix};$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W \quad \text{if and only if} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

In other words

$$W = \text{span}\left(\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}\right).$$

Hence, the set $\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis for W and, therefore, $\dim(W) = 1$. □